



# **INTEGER AND GEOMETRIC OPTIMIZATION IN SAMPLE SURVEYS**

## **ABSTRACT**

**SUBMITTED FOR THE AWARD OF THE DEGREE OF**

**Doctor of Philosophy**

**In**

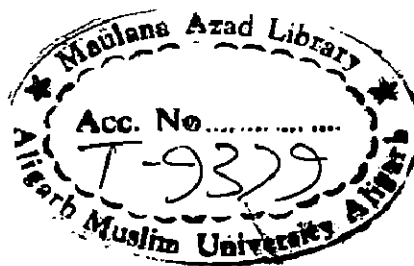
**Operations Research**

**BY**

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**UNDER THE SUPERVISION OF  
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**DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH  
ALIGARH MUSLIM UNIVERSITY  
ALIGARH-202002 (INDIA)  
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## ABSTRACT

This thesis entitled "Integer and Geometric Optimization in Sample Surveys" is submitted to Aligarh Muslim University, Aligarh, India, to supplicate the degree of Doctor of Philosophy in Operations Research. It embodies the research work carried out by me in the Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh.

The technique of obtaining best possible results under any given circumstances is called optimization. The systematic approach to decision making (optimization) generally involves three closely interrelated stages. The first stage towards optimization is to express the desired benefits or the required efforts through other relevant data as a function of certain variables that may be called "decision variables". The second stage continues the process with an analysis of the mathematical model and selection of appropriate numerical technique for finding the optimal solution. The third stage consists of finding an optimal solution, generally with the help of computer.

Any decision problem (with an objective to be maximized or minimized) in which the decision variables must assume non fractional or discrete values may be classified as an integer programming problem. In general, in an integer programming problem the functions representing the objective and constraints may be linear or non linear. An integer programming problem is classified as linear if by relaxing the integer restriction on the variables, the resulting functions are strictly linear. An approach for integer programming problem is the branch and bound technique. This involves setting up of a tree of linear programming problems. The root of the tree is the problem in which all integrality constraints are ignored. If any integer variable is fractional at the optimal solution to this problem, then one considers one such variable say,  $i^{th}$  and partition it as  $x_i = n_{i0} + f_{i0}$  where  $n_{i0}$  is an integer and  $f_{i0}$  is a fractional. Then we set up two alternative problems. One contains the additional constraint  $x_i \leq n_{i0}$  and the other contains the additional constraint  $x_i \geq n_{i0} + 1$ , these constraints can be thought of as branches of a tree. One can, then, form sub-branches on the values of other variables that must be integers. The problem is solved when one has a feasible optimal solution on one branch of the tree and lower values of objective

function on all other branches, whether feasible or not. This important approach, called the 'branch and bound' technique, for solving the all-integer, mixed integer and 0-1 integer problem is originated from the straight forward idea of enumerating implicitly all feasible integer solutions. The initial general algorithm for solving 'all integer' and 'mixed integer' linear programming problem was developed by Land and Doig (1960). This approach for solving integer linear programming problems has been exploited for developing the algorithms in the subsequent chapters.

Geometric programming (GP) is a smooth, systematic and an effective method for solving the non linear class of mathematical programming problems that tends to appear mainly in engineering designs, sampling designs, transportation, planning, economics etc. Clarence Zener, Director of science of Westinghouse in Pittsburgh, Pennsylvania, USA is credited as being the father of GP. Duffin and Zener have developed first the solution procedure of GP for solving an engineering design problem in the early 1960s. It was further extended by Duffin, Peterson and Zener (1967). Geometric programs are not (in general) convex optimization problems, but they can be transformed to convex problems by a change of variables and a transformation of the objective and constraints functions. The convex programming problems occurring in GP are generally represented by an exponential or power function. GP is one of the better tools that can be used to achieve the gain in precision of the estimates with minimal cost. GP can be used not only to provide solution to a problem, but it also can in many instances give a general solution with specific design relationships. These design relationships based upon the design constants can, then, be used for obtaining the optimal solution without having to solve the original problem. This fascinating characteristic appears to be unique in GP. GP may also be considered as a mathematical programming technique for optimizing positive polynomials, which are called 'posynomials'. We have made use of this technique in solving the problems discussed in this thesis.

The work of the thesis is divided into six chapters as follows:

**Chapter 1** is the introductory chapter. It provides an introduction to optimization, unconstrained optimization, constrained optimization, linear and non linear problems, integer programming and its applications in various fields, geometric programming and its applications in various emerging fields, multi-objective programming, bilevel programming and its uses, goal programming and its applications, fuzzy programming

and fuzzy goal programming and their applications in different fields; stochastic programming, chance constrained programming, sampling and census, stratified random sampling, multivariate stratified sampling, double sampling, two and three stage sampling, non-response in sample surveys and randomized response in sample surveys.

The numerical illustration given at the end of the chapters of this thesis is solved by using the optimization software LINGO.

In Chapter 2 we consider a multivariate stratified population with unknown strata weights. An optimum sampling design is proposed in the presence of non-response to estimate the unknown population means using double stratified sampling (DSS) strategy. The problem is formulated as a multi-objective integer nonlinear programming problem (MOINLPP). The problem turns out to be a non-linear bi-level programming problem. Then a fuzzy goal programming approach is used to solve the non-linear bi-level programming problem. The objective function at each level is non-linear in nature and there is one linear constraint with some upper and lower bounds. A compromise optimum allocation has been obtained in the minimum number of steps. The work of this chapter is based on my research paper published in "International Journal of Scientific & Engineering Research" (FRANCE).

Chapter 3 provides the description of a multivariate stratified sampling problem with non linear objective function and a probabilistic non linear cost constraint. The problem is formulated as a multi-objective non-linear programming problem (MONLPP). The fuzzy goal programming (FGP) approach has been used to solve the stochastic multivariate stratified sampling problem with non-linear objective function and probabilistic non-linear cost constraint. In the model formulation of the problem, we first determine the individual best solution of the objective functions subject to the system constraints and construct the non-linear membership functions of each objective. The non linear membership functions are then transformed into equivalent linear membership functions by first order Taylor series at the individual best solution point. Fuzzy goal programming approach is then used to achieve maximum degree of each of the membership goals by minimizing negative deviational variables and finally obtain the compromise allocation. The work of this chapter is based on my

research paper published in the Journal "The South Pacific Journal of Natural and Applied Sciences" (CISRO, FIJI).

In **Chapter 4** an optimum allocation in three-stage sampling design with three variables is considered. Presently, the solution of mathematical programming problems, sampling problems, engineering problems and management problems etc. are very much dependent upon the efficiency of the software. With this regard, we have provided an effective method being handled manually and also by using LINGO software for obtaining optimum allocations in three-stage sampling with the help of primal-dual relationship of GP. The manual description of the solution procedure of GP is very simple in comparison to the complex analytical techniques used in statistical literature. There may not be precise knowledge of parameters in the GP in real world due to insufficient information. The feasibility and effectiveness of the present approach has been illustrated by numerical example. GP optimization technique can be utilized in double sampling design having multiple characters; this is the wider application of the proposed approach. The work of this chapter is based on my research paper published in "International Journal of Scientific & Engineering Research" (FRANCE).

**Chapter 5** provides the use of fuzzy programming for solving a multi-objective geometric programming problem (MOGPP). The problem of non-response with significant travel costs where the cost is quadratic in  $\sqrt{n_h}$  in multivariate stratified sample surveys is formulated as an MOGPP. The fuzzy programming approach is described for solving the formulated MOGPP. The formulated MOGPP is solved with the help of LINGO software and the dual solution is obtained. The optimum allocation of sample sizes of respondents and non-respondents are obtained with the help of dual solution and primal-dual relationship theorem. An illustrative numerical example is given to ascertain the practical utility of the proposed method in sample survey problems in the presence of non-response. The work of this chapter is based on my research paper published in "American Journal of Operations Research".

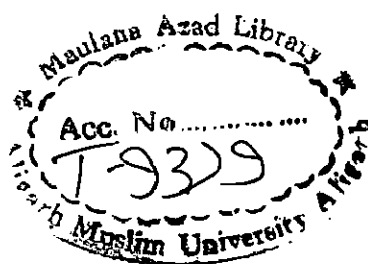
In **Chapter 6** the two-stage randomized response (RR) model in multivariate stratified sample surveys is considered. The problem is formulated as a multi-objective

nonlinear programming problem. A complete method of solution of the formulated problem is projected to solve the problem. A numerical example is worked out to illustrate the computational details of the proposed method. In the next part of this chapter the two-stage stratified Warner's randomized response (RR) model with travel cost is considered and fuzzy geometric programming approach is used to obtain the optimum allocations of sample sizes. The chances of non-response in a multivariate stratified sample survey when the sampling is done with the sensitive questions may be significantly high. To ascertain the practical utility of the fuzzy geometric programming approach in sample surveys problem of randomized response model is extended for multiple sensitive questions to illustrate the fuzzy geometric programming procedure. The work of this chapter is based on my research paper communicated in "Journal of Mathematical Modeling and Algorithms in Operations Research".

A comprehensive list of references, arranged in alphabetical order, is provided at the end of the thesis.

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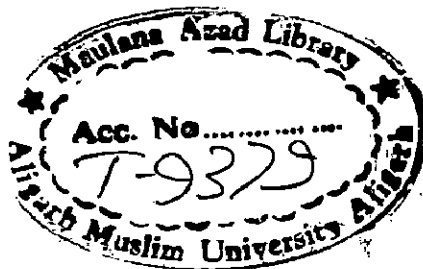
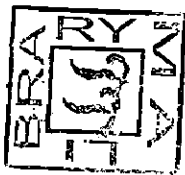
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*Dedicated*  
*To My*  
*Beloved Parents*

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# **CHAPTER – I**

## **Introduction**

## 1.1 OPTIMIZATION

The technique of obtaining best possible results under any given circumstances is called optimization. The systematic approach to decision making or optimization generally involves three closely interrelated stages. The first stage towards optimization is to express the desired benefits, the required efforts and collecting the information as a function of certain variables that may be called “decision variables”. The second stage is the analysis of the mathematical model of the problem and selection of appropriate numerical technique for solving it. The third stage consists of finding an optimal solution, generally with the help of optimization software.

The existence of optimization problem can be traced to the middle of eighteenth century. The work of Newton, Lagrange and Cauchy in solving certain types of optimization problems arising in physics and geometry by using differential calculus methods and calculus of variations is pioneering. These optimization techniques are known as classical optimization techniques and can be generalized to handle cases in which the variables are required to be non-negative and constraints may be inequalities, but these generalizations are primarily of theoretical value and do not usually constitute computational procedures. However, in some simple situations, they can provide solutions, which are practically acceptable.

The optimization problems that have been posed and solved in the recent years have tended to become more and more elaborate, not to say abstract with the development of simplex method by Dantzig in 1947. The necessary and sufficiency conditions for the optimal solution of programming problems worked out by Kuhn and Tucker in 1951 laid the foundations for a great deal of later research in nonlinear programming. The most outstanding example of the rapid development of the optimization techniques occurred with the introduction of dynamic programming by Bellman in 1957 and of the maximum principle by Pontryagin in 1958 these techniques were designed to solve the problems of the optimal control of dynamic systems.

The valuable contributions of Zoutendijk and Rosen to nonlinear programming, Gomory to integer programming and Duffin, Zener, and Peterson to GP during the early 1960s were the most exciting and rapidly developing areas of optimization.

The simply stated problem of maximizing or minimizing a given function of several variables attracted the attention of many mathematicians over the past sixty years or



so for developing the solution techniques under mathematical programming. There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems.

### 1.1.1 Unconstrained Optimization

When the circumstances impose no restrictions on the decision variables, there are no constraints in the optimization problem then the problem is termed as an unconstrained optimization problem. Thus in unconstrained optimization the minimum or maximum of a function of one or more decision variables is sought irrespective of the circumstances. An unconstrained optimization problem may be expressed as:

Find the n-component vector  $\underline{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix} \in E^n$  (the n-dimensional Euclidean space),

such that the function  $f(\underline{x}): E^n \rightarrow E^1$  attains its minimum (or maximum) value at  $\underline{x}^*$ .

- **Local Minimum:** Let the function  $f(\underline{x})$  be defined at all points in some  $\delta > 0$  neighborhood of the point if there exist an  $\varepsilon, 0 < \varepsilon < \delta$  such that  $f(\underline{x}) \geq f(\underline{x}_0)$  for all  $\underline{x}$  in the  $\varepsilon$  neighborhood of  $\underline{x}_0$ .
- **Global Minimum:** Let the function be defined over a closed set  $X \subset E^n$ . The function is said to take on its minimum over  $X$  at the point  $\underline{x}^* \in X$  if  $f(\underline{x}) \geq f(\underline{x}^*) \forall \underline{x} \in X$ .

In unconstrained optimization the aim is to find global optimum (maximum or minimum) of a given objective function consisting of one or more variables. As minimization of  $f(\underline{x})$  is equivalent to maximization of  $-f(\underline{x})$ . The basic philosophy of most of the numerical methods of unconstrained minimization is to produce a sequence of improved approximations to the minimum according to the following scheme.

Step 1 Start with an initial trial point  $\underline{x}_1 \in E^n$ .

Step 2 Find a suitable direction  $\underline{s}_i$  ( $i = 1$  to start with) which points in the direction of the minimum.

Step 3 Find the minimizing step length  $\lambda_i^*$  along the direction  $\underline{s}_i$ .

Step 4 Obtain the next trial point

$$\underline{x}_{i+1} = \underline{x}_i + \lambda_i^* \underline{s}_i$$

Step 5 Test whether  $\underline{x}_{i+1}$  is the required optimum. If  $\underline{x}_{i+1}$  is optimum, put  $\underline{x}_i^* = \underline{x}_{i+1}$  and Stop; Otherwise, set  $i = i + 1$ , and go to step (ii).

In this thesis mainly minimization methods have been discussed. Some of the methods for finding unconstrained optimum of the univariate or multivariate functions are as given below:

- Fibonacci method
- Golden section method
- Quadratic interpolation method
- Cubic interpolation method
- Decent or Gradient method
  - Steepest Descent Method
  - Conjugate Gradient Method
  - Variable Metric Method

Cauchy made the first application of the steepest decent method to solve unconstrained minimization problems. In spite of these early contributions, very little progress was made until the middle of the twentieth century, when high-speed digital computers made the implementation of the optimization procedures easy and stimulated further on new methods. The major developments in the area of numerical methods of unconstrained optimization have been made in United Kingdom. McCormick solved many difficult problems by using the well-known techniques of unconstrained optimization.

### 1.1.2 Constrained Optimization:

When the decision variables of an optimization problem are restricted to follow certain relations called constraints in the form of inequalities or equations the problem is called a constrained optimization problem.

The constrained optimization problems are popularly known as “mathematical programming problem (MPP).” Thus, in this thesis a constrained optimization problem is referred to as a (MPP). Mathematically, it can be stated as:

$$\left. \begin{array}{l} \max(\text{or min}) Z = f(\underline{x}) \quad (i) \\ \text{subject to } g_i(\underline{x}) \{ \leq \text{ or } = \text{ or } \geq \} \underline{b}_i, i = 1, 2, \dots, m \quad (ii) \\ \text{and } \underline{x} \geq \underline{0}. \end{array} \right\} \quad (1.1)$$

- **A Feasible Solution:** An  $n$ -component vector  $\underline{x}$  is called a feasible solution to the MPP (1.1) if it satisfies  $g_i(\underline{x}) \{ \leq \text{ or } = \text{ or } \geq \} \underline{b}_i, i = 1, 2, \dots, m$ . The set  $F$  of all feasible solutions to the MPP (1.1) is defined as:

$$F = \{ \underline{x} \mid g_i(\underline{x}) \{ \leq \text{ or } = \text{ or } \geq \} \underline{b}_i, i = 1, 2, \dots, m, \underline{x} \geq \underline{0} \}$$

- **An Optimal Solution:** An  $\underline{x}^* \in F$  will be the optimal solution to the MPP(1.1) if  $f(\underline{x}) \geq f(\underline{x}^*)$  in case of minimization and  $f(\underline{x}) \leq f(\underline{x}^*)$  in case of maximization  $\underline{x} \in F$ .

Mathematical Programming (MPP) can broadly be classified into two disjoint classes.

- **Linear Programming Problems:** When in an MPP all the involved functions are linear it is called a linear programming problem (LPP).
- **Non-linear Programming Problems:** When at least one of the involved functions is non-linear or in other words when all the involved functions are not linear then the MPP is called a non-linear programming problem (NLPP).

Some of the MPP used in this thesis are discussed below:

## 1.2 INTEGER PROGRAMMING (IP)

Any decision problem (with an objective to be maximized or minimized) in which the decision variables must assume non fractional or discrete values may be classified as an integer programming problem (IPP). In general, in IPP the functions representing the objective and constraints may be linear or non linear. An IPP is classified as linear if by relaxing the integer restriction on the variables, the resulting functions are strictly linear. The general mathematical model of an IPP can be stated as:

$$\left. \begin{array}{l} \max (\text{or min}) Z = f(\underline{x}) \\ \text{subject to } g_i(\underline{x}) \{ \leq \text{ or } = \text{ or } \geq \} b_i, i = 1, 2, \dots, m \\ x_j \geq 0, \quad j = 1, 2, \dots, n. \\ x_j \text{ is an integer for } j \in J \subseteq I = (1, 2, \dots, n) \end{array} \right\} \quad (1.2)$$

where  $\underline{x} = (x_1, \dots, x_n)$  is  $n$ -component vector of decision variables

If  $J = I$ , that is, all the variables are restricted to be integers we have an all (or pure) integer programming problem (AIPP) Otherwise, if  $J \subset I$ , i.e., not all the variables are restricted to be integers, we have a mixed integer programming problem (MIPP).

In most of the practical situations the values of the decision variables are required to be integers. Cutting plane methods are the first systematic technique to be developed for the IPPs. The early works of Dantzig *et al.* (1954) and Markowitz and Manne (1957) directed the researchers for solving the LPPs in integers. Dantzig (1958, 1959) discussed the integer solutions to some special LPPs. Gomory (1958) suggested the first systematic method to obtain an optimal integer solution to an AIPP. Later, Gomory (1960, (1963) extended the method to deal with the more complicated case of MIPP, when only some of the variables are required to be integers. A primal algorithm was first introduced by Ben-Israel and Charnes (1962), but Young (1965) was the first to develop a finite primal algorithm. Also, Harris (1964) developed a mixed-integer programming using Benders's (1962) partitioning scheme. Trotter and shetty (1974) proposed an algorithm for the bounded variable of AIPPs. Granat and Granat (1977) constructed a new cutting plane algorithm for solving the integer fractional programming and mixed integer fractional programming problems with the help of Charnes and Cooper's (1962). Khan and Bari (1977) developed a procedure for integer solution to some allocation problems. Balas *et al* (1996) has developed revisited Gomory's cut, Magnanti and Vachani (1990) have given strong cutting plane algorithm for production scheduling with changeover costs. Bari and Ahmad (2003) has developed a procedure of Naz-Cut for IP. Bari and Alam (2005) has developed At-Cut.

Another approach for IPP is the branch and bound technique. This involves setting up of a tree of LPPs. The root of the tree is the problem in which all integrality constraints are ignored. If any integer variable is fractional at the optimal solution to this problem, then one considers one such variable, say  $i^{th}$  and partition it as

$x_i = n_{i0} + f_{i0}$  where  $n_{i0}$  is an integer and  $f_{i0}$  is a fractional. Then we set up two alternative problems. One contains the additional constraints  $x_i \leq n_{i0}$  and the other contains the additional constraints  $x_i \geq n_{i0} + 1$ , these constraints can be thought of as branches of the tree. One can, then, form sub-branches on the values of other variables that must be integers. The problem is solved when one has a feasible optimal solution on one branch of the tree and lower values of objective function on all other branches, whether feasible or not. An important approach called the 'branch and bound' technique for solving the all-integer, mixed integer and 0-1 integer problems have originated the straight forward idea of enumerating implicitly all feasible integer solutions. A general algorithm for solving 'all integer' and 'mixed integer' LPP was developed by A.H. Land and A.G. Doig (1960). Later Dakin (1965) proposed another interesting variation of Land and Doig algorithm. Bertier and Roy (1965) presented a general theory for branching and bounding. Also, Balas (1965) introduced an interesting enumerative algorithm for LPP with the variables having the value zero or one and named it as zero-one programming problem. Balas (1968) restated their theory in simpler form. Vinod (1969) has discussed IP and the theory of grouping. Latter Mitten (1970) generalized and slightly extended Balas work. Bari and Arshad (1978) has developed a branch and bound method for integer quadratic programming.

An important and widespread area of application of IP concerns the management and the efficient use of scarce resources to increase productivity. These applications include operational problems such as the distribution of goods, production scheduling, and machine sequencing. They also include planning problems such as capital budgeting, facility location, portfolio analysis, economic optimization, strategic planning and design problems such as communication and transport network design, circuit design and the design of automated production systems. IP has also found application in sectoral or single-industry planning models, for example, Manne (1967) applied IP technique to several industries of Indian economy and Westhal (1971) applied mixed integer programming to the economy of South Korea. Liittschwager and Wang (1978) have discussed IP solution of a classification problem. Arabinda (1984) has discussed school timetabling case in large binary integer linear programming. Mathur *et al.* (1985) discussed some of the problem faced by a decision

maker in managing a communication network. They applied integer programming to assign frequencies to the different nets of network so that inter modulation interfaces are within desirable limits. Dowman and Rangel (1998) investigated properties of IP models for a class of production planning problems. Wan and Lo (2003) presented a mixed integer formulation for multiple-route transit network design problem. Tao *et al.* (2004) proposed mixed integer programming to maximize the number of infected women cured of C-trachomatis infection. Avella and Vasil (2005) described a case study where branch-and-cut algorithm yields optimal solution of a real-world timetabling problem of university courses. A lot of work has also been done on integer and MIPP. Few of them are due to Guignard and Spielberg (1981), Lenstra (1983), Hoffman and Padberg (1991), Savelsberg (1994), Weismantel (1997), Patel and Chinneck (2007), Mc Lay and Jacobson (2007), Cadoux (2008), Al-Yakoob *et al.* (2010). Alam and Bari (2004) proposed a new approach for solving mixed integer linear programming problem. Ahmad and Bari (2004) proposed a branch and bound procedure to solve n-jobs and 3-machines flow-shop scheduling problems with transportation time. Alam and Bari (2004) proposed optimality criteria for IPPs. Alam and Bari (2004) developed a heuristic approach for solving multi-objective linear IPP. Kesavan *et al.* (2004) and Rabbani *et al.* (2007) have developed integer optimum through Modified Fourier's Method. Bakir and Aksop (2008) have discussed a 0-1 IP approach to a university timetabling problem programming. Paul (2009) has written a book on Logic and IP. Ali *et al.* (2011) have used integer goal programming approach in reliability allocation problem. Raghav *et al.* (2011) have used goal programming and Lexicographic goal programming approaches in bi-objective stratified sampling and obtained an integer solution. Ali *et al.* (2011) have obtained an integer solution in allocating repairable and replaceable components for a system availability using selective maintenance. Semiz (2004) used goal programming to provide a flexible approach to obtain feasible and efficient integer compromise allocations. Diaz-Garcia and Cortez (2006) addressed the problem of optimum allocation in multivariate stratified sampling as a programme of multi-objective optimization of integers under some prior information about the population. Khare (1987) discussed the problem of optimum allocation in univariate stratified sampling design in presence of non-response whereas Varshney *et al.* (2011, 2012b) worked out optimum allocation in presence of non-response for multivariate double sampling for stratification. The problem is formulated as a multi-objective integer nonlinear programming with

bounded variables and used goal programming to develop a procedure for its solution. Khowaja *et al.* (2011) treated the problem of obtaining a compromise allocation  $n_h$  as an AINLPP. They worked out an explicit formula for  $n_h$  using Lagrange multipliers technique. They showed through a numerical example that the formula works well and the rounded off values of  $n_h$  provide a nearly optimal values of  $n_h$ . However, if this is not the case, they suggested using the formula to obtain an initial point to start a cutting plane technique to obtain an integer solution. Raghav *et al.* (2011) have also developed a new cutting plane method for finding the integer solution of allocating problem in stratified sampling.

### 1.3 GEOMETRIC PROGRAMMING

Geometric programming (GP) is a smooth, systematic and an effective method for solving the non linear class of MPPs that tends to appear mainly in engineering design, sampling design, transportation, planning, economics etc. Clarence Zener, Director of science of Westinghouse electric in Pittsburgh, Pennsylvania, USA is credited as being the father of GP. Duffin and Zener have first developed the solution procedure of GP for solving engineering design problem in the early 1960s and further extended by Duffin, Peterson and Zener (1967). Geometric programs are not (in general) convex optimization problems, but they can be transformed to convex problems by change of variables and a transformation of the objective and constraints functions. The convex programming problems occurring in GP are generally represented by an exponential or power function. GP is one of the better tools that can be used to achieve the gain in precision of the estimates with minimal cost. GP can be used not only to provide solution to a problem, but it also can in many instances give a general solution with specific design relationships. These design relationships based upon the design constants can, then, be used for the optimal solution without having to solve the original problem. This fascinating characteristic appears to be unique in GP. GP is a mathematical programming technique for optimizing positive polynomials, which are called posynomials. Posynomials are not the same as polynomials in several independent variables. Polynomial's exponents must be non-negative integers, but its independent variables and coefficients can be arbitrary real numbers; on the other hand, a posynomial's exponents can be arbitrary real numbers, but its independent variables and coefficients must be positive real numbers.

Posynomials are closed under addition, multiplication and nonnegative scaling. This terminology was introduced by Richard J. Duffin, Elmor L. Peterson and Clarence in 1963 in their seminal book on GP. These functions are also known as "posinomials" in some literature. Posynomial functions are minimized in the GP technique subject to several constraints. Posynomial functions can be defined as polynomials in several variables with positive coefficients in all terms and the power to which the variables are raised can be any real numbers. GP always transforms the primal problem of minimizing a posynomial subject to posynomial constraints to a dual problem of maximizing a function of the weights on each constraint. This technique has many similarities with respect to linear programming but also has advantages in sense that:

- a non-linear objective function can be used;
- the constraints can also be non-linear;
- the optimal values of the costs, gain in precision of the estimates exectra can be determined with the dual without first determining the specific values of the primal variables.

GP can lead to generalized design solutions and specific relationships between variables. The relationships can be determined in generalized term such as costs, gain in precision of the estimates etc., when the degrees of difficulty are low such as zero or one. The major disadvantage is that the mathematical formulation is much more complex than linear programming and hence the complex problems are very difficult to solve. The degree of difficulty (DD) plays a significant role for solving a non-linear programming problem by GP method. The degree of difficulty of a GP problem is defined as:

$$\text{Degree of difficulty} = \text{total number of terms in the objective functions and constraints} - \text{total number of decision variables of the objective functions and constraints} - 1$$

If the degree of difficulty of primal problem is zero, then unique dual feasible solution exists. If the problem has positive degree of difficulty, then the objective function can be maximized by finding the dual feasible region, and if there is negative degree of difficulty then inconsistency of the dual constraints may occur. The dual of a GP problem with negative degree of difficulty is often infeasible. It has been suggested that such problems can be solved by finding a dual "approximate" solution which



minimizes a measure of the infeasibility, for e.g., the summed squares of the infeasibilities in the dual constraints.

It is called GP because it is based upon the arithmetic-geometric inequality where the arithmetic mean is always greater than or equal to geometric mean. That is

$$\frac{(x_1 + x_2 + \dots + x_n)}{n} \geq (x_1 * x_2 * \dots * x_n)^{\frac{1}{n}}$$

### 1.3.1 Geometric Programming in Sample Surveys

In case of sample surveys, the mathematical form of GP can be given as follows:

Primal form of GP

$$\left. \begin{array}{l} \min f_0(x) \\ \text{subject to } f_k(x) \leq 1, \quad k = 1, 2, \dots, p, \\ \text{and } x_i > 0, \quad i = 1, 2, \dots, m \end{array} \right\} \quad (1.3)$$

$$f_k(x) = \sum_{i \in J[k]} c_i t_1^{a_{i1}} t_2^{a_{i2}} \dots t_m^{a_{im}}, \quad k = 1, 2, \dots, p$$

where  $a_{ij}$ : arbitrary real numbers,  $c_i$ : positive,  $f_k(x)$ : posynomials.

Dual form of GP:

$$\left. \begin{array}{l} \max v(w) = \left[ \prod_{i=1}^m \left( \frac{c_i}{w_i} \right)^{w_i} \right] \prod_{k=1}^p \lambda_k(w)^{\lambda_k(w)} \\ \text{subject to } \sum_{i \in J[0]} w_i = 1 \quad \text{Normality condition} \\ \sum_{i=1}^m a_{ij} w_i = 0, \quad j = 1, 2, \dots, p \quad \text{Orthogonality condition} \\ w_i \geq 0 \quad \text{Positivity condition} \\ \lambda_k(w) = \sum_{i \in J[k]} w_i, \quad i = 1, 2, \dots, m, \quad k = 1, 2, \dots, p. \end{array} \right\} \quad (1.4)$$

Primal-dual relationship theorem: If  $w_{0i}^*$  is a maximizing point for dual problem (1.4), each minimizing points  $(x^*)$  for primal problem (1.3) satisfies the system of equations:

$$f_{0j}(n) = \begin{cases} w_{0i}^* v(w^*), & i \in J[0] \\ \frac{w_{ij}}{v_L(w_{0i}^*)}, & i \in J[L] \end{cases} \quad (1.5)$$

where  $L$  ranges over all positive integers for which  $v_L(w_{0i}^*) > 0$ .

The optimal values of sample sizes ( $x^*$ ) can be calculated with the help of the primal – dual relationship theorem (1.7).

The first paper on GP in optimizing engineering designs was published by Zener in 1961. The basic concepts on GP and its applications were discussed by many authors. They are as follows: Beightler and Phillips (1976), Sengupta (1977), Charnes *et al.* (1977), Ecker (1980), Avriel (1980), Feigin and Passy (1981), Rijckaert (1985), Chen (1992), Kortanek and Hoon (1992), Yeh (1993), Maranas and Floudas (1997), Mutapcic *et al.* (2006), Stephen *et al.* (2007), Many authors have used GP for solving engineering design problems. Some of them are as follows: Avriel and William (1970), Duffin (1970), Zener (1971), Duffin and Peterson (1973), Chu and Wong (2001b), Dawson *et al.* (2001), Hershenson *et al.* (2001), Mandal and Visvanathan (2001), Peterson (2001), Hershenson (2002), Colleran *et al.* (2003), Chiang and Boyd (2004), Kim *et al.* (2004), Vanderhaegen and Brodersen (2004), Boyd *et al.* (2005), Jabr (2005), Joshi *et al.* (2005), Shiang (2007), Shaojian *et al.* (2008).

Ahmad and Bohman (1987) have discussed an application of GP in multivariate double sampling. Maqbool *et al.* (2011) have discussed the GP approach to find the optimum allocations in multivariate two-stage sample surveys. Several authors have discussed planning of industrial complexes, transportation planning, optimal designs, sensitivity analysis, quality control, inventory etc. by means of GP. Among them are as: Passey (1972), Nijhamp (1972), Ecker and Zorachi (1976), Jefferson and Scott (1978), Dembo (1982), Smeers and Tyteca (1984), Davis and Rudolph (1987), Devis and Robert (1989), Kyparsis (1990), Illes (1991), Hariri and Abou-El-Ata (1997), Yun (1997), Xi (1997), Islam and Roy (2005), El-Ata *et al.* (2003), Scott *et al.* (2004).

Maximum likelihood, entropy maximization and the GP approaches to the calibration of trip distribution models was given by Wong (1981), Wall *et al.* (1986) have solved complex chemical equilibrium problem using a GP. Hajela (1986) has discussed GP strategies for large-scale structural synthesis. Klafszky *et al.* (1992) have discussed

GP approach to the channel capacity problem. Choi and Bricker (1996a) have discussed effectiveness of a GP algorithm for optimization of machining economics models. Bricker *et al.* (1997) have discussed a GP approach in maximum likelihood estimates with order restrictions on probabilities and odd ratios, Sonmez *et al.* (1999) have discussed dynamic optimization of multipass milling operations via GP. Cheng *et al.* (2002, 2005) have used GP approach in univariate cubic *LI* splines. Alexandre *et al.* (2004) have discussed a general alternative procedure for solving negative degree of difficulty problems in GP; Yazarel and Pappas (2004) have used GP relaxations for linear system reachability. Singh *et al.* (2005) have given robust gate sizing by GP; Chiang (2005b) has used GP for communication systems. Hsiung *et al.* (2006) have discussed tractable approximate robust GP. Multi-objective geometric programming (MOGP) problem was discussed by Ojha and Biswal (2010), Ojha and Das (2010) and Islam (2010) in different fields. Jitka Dupačová (2010) has discussed the stochastic geometric programming (SGP) with an application. Ghosh and Roy (2013) have described goal geometric programming (GGP) problem with crisp and imprecise targets.

#### 1.4 MULTI-OBJECTIVE PROGRAMMING

The developments of simplex method by Dantzig for solving linear programming problems, various aspects of single objective mathematical programming have been studied quite extensively. It was, however, realized that almost every real life problem involves more than one objective. Multi-objective programming is a powerful mathematical procedure and applicable in decision making to a wide range of problems in the government organizations, non profitable organizations and private sector etc.

A multiple objective multi-objective programming linear programming model with objective functions can be stated as follows:

$$\left. \begin{array}{l} \max(\min): \{f_1(X), f_2(X), \dots, f_p(X)\} \\ \text{subject to } X \in S \end{array} \right\} \quad (1.6)$$

where  $f_i(X) \forall i = 1, 2, \dots, p$  is a linear function of decision variable  $X$  and  $S$  is the set of feasible solutions. The ideal solution for a multiple objective linear programming problem would be to find that feasible set of decision variables  $X$  which would optimize the individual objective function of the problem simultaneously. However,

with the conflicting objectives in the models, a feasible solution that optimizes one objective may not optimize any of the other remaining objective functions. This means that what is optimal in terms of one of the  $p$  objectives is generally not optimal for the other  $p - 1$  objectives i.e. multiple objective optimization has none in which we may optimize all the objectives simultaneously.

A number of methodologies have been developed to handle the problem of multiple objectives. Methods of multi-objective optimization can be classified in many ways according to criteria. They are categorized into two relatively distinct subsets: generating methods and preference based methods. In generating methods, the set of pareto optimal (or efficient) solutions is generated for the decision maker who, then, chooses one of the alternatives. In preference based methods, the preference of the decision maker are taken into consideration as the solution process goes on, and the solution that best satisfies the decision maker's preferences is selected. In fact there is no universally accepted definition of "optimum" in multiple objective optimizations as it is in single objective optimization, which makes it difficult to even compare results of one method with another. Normally, the decision about what the "best answer is" corresponds to the so-called human decision maker. Coello (1999) has done a comprehensive survey in this field.

## **1.5 BILEVEL PROGRAMMING**

Bilevel programming problems (BLPP) have been introduced to the optimization community in the seventies of the 20<sup>th</sup> century, although its first formulation dates back to 1934 when it has been formulated by H.V. Stackelberg in a monograph on market economy. The original formulation for the bilevel programming appeared in a paper authored by Bracken and McGill (1973), although it was Candler and Norton (1977) who, for the first time, used the term bilevel and multilevel programming. Motivated by the game theory of Stackelberg, several authors studied bilevel programming intensively and contributed to its proliferation in the mathematical programming community. A sequential optimization problem in which independent decision makers act in a non cooperative manner to maximize their individual benefits may be categorized as a Stackelberg game. The Stackelberg game is conceptually extended to the multilevel programming problem, in which the players are required to move in turn and the strategy sets are no longer assumed to be disjointed. Here

decision problems involving multiple agents invariably lead to conflict and gaming. Multi-agent systems have been analyzed using approaches that explicitly assign to each agent a unique objective function and set of decision variables. The system is defined by a set of different constraints for each agent. The decisions made by each agent in these approaches affect the decisions made by the others and their objectives. There is a hierarchical ordering of agents and one set has the authority to strongly influence the preferences of the other agents. The final decisions are executed sequentially within the hierarchy, from highest to lowest levels. BLPP is a special case of multilevel programming problem with a structure of two levels, namely, the upper level and the lower level. The upper level decision maker is called the leader and that of the lower level is called the follower. The follower executes its policies after, and in view of, the decisions of upper level decision maker. Control over the decision variables is partitioned among the levels, but a decision variable of one level may affect the objective function of other level. Thus, an important feature of BLPP is that a planner at one level of the hierarchy may have his objective function and decision space determined, in part, by variables controlled at other level. However, his control instruments may allow him to influence the policies at other level and thereby improve his own objective function.

The problems we want to consider have the following common characteristics.

1. The system has interacting decision-making units within a hierarchical structure.
2. The execution of decision is sequential, from upper to lower level. The follower executes its policies after, and in view of, the decisions of the leader.
3. Each decision-making unit optimizes its own objective function independently of other units, but is affected by the actions and reactions of the other unit.
4. The external effect on a decision maker's problem can be reflected in both his objective function and his set of feasible decisions.

Let us consider a bilevel hierarchical system where a vector of decision variables  $(x, y) \in \mathcal{R}^n$  be partitioned among the two, upper and lower level decision makers i.e. leader and follower respectively. The leader has the control over the decision variable  $x \in \mathcal{R}^{n_1}$  and follower over the variable  $y \in \mathcal{R}^{n_2}$ , where  $n_1 + n_2 = n$ . Furthermore,

assuming that  $F, f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^1$  are linear and bounded the linear BLPP can be stated as follows:

$$P1: \max_x F(x, y) = ax + by \text{ where } y \text{ solves}$$

$$P2: \max_y f(x, y) = cx + dy$$

$$\text{subject to } Ax + By \leq r$$

where  $a, c \in \mathbb{R}^{n_1}$ ,  $b, d \in \mathbb{R}^{n_2}$ ,  $r \in \mathbb{R}^m$ ,  $A$  is an  $m \times n_1$  matrix,  $B$  is an  $m \times n_2$  matrix. Let the feasible choices of  $(x, y)$  be denoted by the constraint region

$S = \{(x, y) | Ax + By \leq r\}$ . Hence for each value of  $x$ , lower level will react with a

corresponding value of  $y$ . This induces a functional relationship between the

decisions of leader and the reactions of the follower. For a given  $x$ , let  $Y(x)$  denote

the set of optimal solutions to the inner problem,  $P2$ ,  $\max_{y \in Q(x)} \tilde{f}(y) = dy$  where

$$Q(x) = \{y | By \leq r - Ax\}$$

and represent the upper level decision maker's solution space, or the set of rational reactions of  $f$  over  $S$ , as

$$\psi_f(S) = \{(x, y) \in S, y \in Y(x)\}.$$

We assume that  $S$  and  $Q(x)$  are bounded and non-empty. The definitions of feasibility and optimality for the linear BLPP are given by the following:

*Definition 1.* A point  $(x, y)$  is called feasible if  $(x, y) \in \psi_f(S)$ .

*Definition 2.* A feasible point  $(x^*, y^*)$  is called optimal if  $ax^* + by^*$  is unique for all  $y^* \in Y(x^*)$  and  $ax^* + by^* \geq ax + by$  of all feasible pairs  $(x, y) \in \psi_f(S)$ .

The general linear BLPP is of the following form

$$\max_x F(x, y) = ax + by \text{ where } y \text{ solves}$$

$$\max_y f(x, y) = cx + dy$$

subject to  $Ax + By \leq r$

$$x_1, \dots, x_i \in \mathfrak{R}, \quad y_1, \dots, y_j \in \mathfrak{R}$$

$$x_{i+1}, \dots, x_{n_1} \in Z^+ \quad y_{j+1}, \dots, y_{n_2} \in Z^+$$

where  $x$  is a vector of upper level variables, of which  $i$  are continuous and  $y$  is a vector of lower level variables, of which  $j$  are continuous.

The methods developed for the solution of integer BLPP include a branch and bound type of enumerative solution algorithm for all integer BLPP developed by Moore and Bard (1990). Narula and Nwosu (1983, 1985) also proposed procedure via regular simplex pivots with modifications after taking the dual of the problem P2. The first method using vertex enumeration approach was proposed by Candler and Townsley (1982). Bialas and Karwan (1984) developed a similar vertex enumeration procedure called the *Kth*-best algorithm. For the solution of the mixed integer BLPP, another branch and bound technique is developed by Wen and Yang (1990). Wen *et al.* (1991) have discussed linear BLPP. Bard and Moore (1992) also presented a new algorithm for binary variables. Cutting plane and parametric solution approaches have been developed by Dempe (1995). Jan and Chern (1994) proposed an algorithm using parametric analysis to solve nonlinear mixed integer BLPP. Shi *et al.* (2005a) extended *Kth*-best approach and also applied it for linear bilevel multi-follower programming problem (Shi *et al.* 2005b). Also Sahin and Ciric (1998), Zeynep and Floudas (2005) developed algorithms for these type of problem. Adhami *et al.* (2009) has developed an algorithm for the mixed-integer concave quadratic BLPP. Adhami *et al.* (2009) has developed an extended *Kth*- best approach for solving integer BLPP. Adhami and Rabbani (2014) have developed a procedure for solving integer BLPP. On the basis of the gradient information generated from the lower level optimization problem, Kolstad and Lasdon (1990) proposed a heuristic descent algorithm for the BLPP. Vincete *et al.* (1994) and Jiye *et al.* (2000) presented a descent method for solving quadratic BLPP have discussed bilevel programming framework for enterprise-wide process networks under uncertainty.

## 1.6 GOAL PROGRAMMING

The goal programming is one of the most important technique for solving multi-objective optimization problems. The GP was firstly introduced and successfully used for solving multi-objective problem by Charnes *et al.* in 1955. Several books on GP with its applications in management, accounting, industrial and decision analysis were written by Charnes and Cooper (1961), Ijiri (1965) and Lee (1972). Later on Ignizio (1983, 1985) described the GP in simplest way. In GP our objective is to optimize the aspiration levels and deviations from it. The goal will be formed by an objective function along with an aspiration level. We denote the aspiration level of the objective function  $f_i$  by  $z_i$  for  $i = 1, 2, \dots, k$ . For minimization problem goals are of the form  $f_i \leq z_i$  and for maximization problem  $f_i \geq z_i$ . Goals are also in the form of equalities or ranges (Charnes and Cooper (1961)). In developing a GP model the decision variables of the model are to be defined first. Then the goals related to the decision variables are to be listed in order of priority. When the aspiration level have been specified, the following task is to minimize the under and over achievements of the objective function values with respect to the aspiration levels. The study of deviational variables  $\delta_i = z_i - f_i(x)$  is sufficient for it. The value of the deviational variable  $\delta_i$  depends upon the problem whether it is positive or negative. It is defined as the difference of two positive variables  $\delta_i = \delta_i^- - \delta_i^+$ . It has been investigated that the attainment of the aspiration level depends upon the deviational variables. We can write  $f_i(x) + \delta_i^- - \delta_i^+ = z_i$  for all  $i = 1, 2, \dots, k$ , where we have to minimize the sum of deviational variables (Charnes and Cooper (1977)). It means that along with the aspiration levels the decision maker must specify information about the importance of attaining the aspiration level in the form of weighting coefficients. The weighting coefficients are assumed to be positive and their sum will be up to one. The bigger the weighting coefficient is, the more it is important for attainment of aspiration level. (Sometimes negative coefficients are used to represent a premium instead of penalty.)

The mathematical form of weighted GP problem is as follows

$$\left. \begin{array}{l} \min \sum_{i=1}^k w_i |f_i(x) - z_i| \\ \text{subject to } x \in s \end{array} \right\} \quad (1.7)$$



Since it may not always be possible to achieve each and every goal fully a lexicographic procedure is used in which the various goals are satisfied in order of their relative importance.

$$\delta_i^+ = \max[0, f_i(x) - z_i] \text{ or } \delta_i^- = \frac{1}{2} [z_i - f_i(x) + f_i(x) - z_i]$$

and under achievement variables

$$\delta_i^- = \max[0, z_i - f_i(x)] \text{ or } \delta_i^+ = \frac{1}{2} [z_i - f_i(x) + z_i - f_i(x)]$$

Therefore the resulting weighted GP problem is

$$\left. \begin{array}{l} \min \sum_{i=1}^k w_i (\delta_i^- + \delta_i^+) \\ \text{subject to } f_i(x) + \delta_i^- - \delta_i^+ = z_i \quad \forall i = 1, 2, \dots, k \\ \delta_i^-, \delta_i^+ \geq 0 \quad \forall i = 1, 2, \dots, k \\ x \in S \end{array} \right\} \quad (1.8)$$

A wide survey of literature of GP up to 1983 is described in Soyibo (1985). A survey of GP is also given in Kornbluth (1973) and Ignizio (1976). Further a wide collection of journal, papers and books on GP is assembled in Schniederjans (1995a). Weighted GP with equal weighting coefficients is employed the planning of public works in Yoshikawa *et al.* (1982). Weighted GP with sensitivity analysis is also used for portfolio section in Tami *et al.* (1996).

## 1.7 FUZZY PROGRAMMING (FP)

In some situations the decision makers are unable to think about the appropriate use of the commonly used probability distributions, especially when the information is vague, relating human language and behavior, imprecise or ambiguous system data or when the information could not be described and defined well due to limited knowledge and deficiency in its understanding. Such type of uncertainties can be categorized as fuzziness. A system with vague and ambiguous information can neither be formulated nor solved effectively by traditional mathematics-based on optimization techniques nor probability-based stochastic optimization approaches. However, fuzzy set theory, which was developed by Zadeh in 1960's and FP techniques provide a useful and efficient tool for modeling and optimizing such systems. Modeling and optimization under a fuzzy environment is called modeling

and fuzzy optimization. Hence a mathematical programming problem under fuzzy environment is known as fuzzy mathematical programming problem (FMPP). FMPP is also known as constrained fuzzy optimization problem.

### 1.7.1 Fuzzy Goal Programming

Fuzzy goal programming (FGP) utilizes fuzzy set theory (Zadeh (1965)) to deal with a level of imprecision in the goal programming model. This imprecision normally relates to the goal target values ( $b_q$ ) but could also relate to other aspects of the goal programme such as the priority structure. The early fuzzy goal programming models used both Chebyshev (Narasimhan, 1980; Hannan, 1981) and weighted (Hannan, 1981; Tiwari *et al.* 1987) distance metrics. Mohamed (1997) has given the relationship between goal programming and fuzzy programming; there are various possibilities for measuring the fuzziness around the target goals, each of which leads to a different fuzzy membership function. These functions model the drop in dissatisfaction from a state of total satisfaction (where the membership function takes the value 1) to a state of total dissatisfaction (where the membership function takes the value 0). There are many possible fuzzy membership functions, the algebraic structure of two of the most common linear fuzzy membership functions are outlined below

1. Right-sided (positive deviations penalized) linear function:

$$\mu(f_q(x)) = \begin{cases} 1, & f_q(x) \leq b_q \\ 1 - \frac{f_q(x) - b_q}{p_{\max}}, & b_q \leq f_q(x) \leq b_q + p_{\max} \\ 0, & f_q(x) \geq b_q + p_{\max} \end{cases} \quad (1.9)$$

2. Left-sided (negative deviations penalized) linear function:

$$\mu(f_q(x)) = \begin{cases} 1, & f_q(x) \geq b_q \\ \frac{b_q - f_q(x)}{n_{\max}}, & b_q - n_{\max} \leq f_q(x) \leq b_q \\ 0, & f_q(x) \leq b_q - n_{\max} \end{cases} \quad (1.10)$$

Buckley and Feuring (2000) have discussed an evolutionary algorithm solution to fuzzy problems. Chanas and Kuchta (1998) have discussed the discrete fuzzy

optimization. Various models and approaches to fuzzy linear programming are given by: Dubois (1987), Zimmerman (1991), Drinkov *et al.* (1993) have given an introduction to Fuzzy Control. Wang and Fang (1997) have discussed genetic-based approach for aggregate production planning in fuzzy environment, Fang and Hu (1999) have discussed linear programming with fuzzy coefficients in constraints. Fuzzy multi-objective programming is given by many authors such as: Sakawa and Yano (1989), Sakawa and Yano (1994), fuzzy integer programming is given by Stoica *et al.* (1984), fuzzy dynamic programming is given by Kacprzyk and Esogbue (1996), possibilistic linear programming is given by many authors such as: Ramik and Riinane (1985), Rommelfanger (1989) and fuzzy non-linear programming is given by many authors such as: Tang and Wang (1997), Sakawa and Yano (1989), Tang and Richard (1998), Trappey *et al.* (1998), Liu and Fang (2001), Maleki (2002) have described the ranking functions and their applications to fuzzy linear programming. Wu (2004) has given the duality theory in fuzzy optimization problem. Luo *et al.* (2005) have given an inexact fuzzy two-stage stochastic model for quantifying the efficiency of nonpoint source effluent trading under uncertainty. Nasser *et al.* (2005) have discussed the simplex method for solving linear programming problem with fuzzy numbers. Mahadavi-Amiri and Nasser (2006) described the duality in fuzzy number linear programming by the use of a certain linear ranking function. Ganesan and Veeramani (2006) have described fuzzy linear programs with trapezoidal fuzzy numbers. Mahadavi-Amiri and Nasser (2007) have discussed the duality results and a dual simplex method for linear programming problem with trapezoidal fuzzy variables. Maleki and Mashinchi (2007) have described the multi objective geometric programming (MOGP) with fuzzy parameters. Nasser (2008) described a new method for solving fuzzy linear programming problem by linear programming. Nasser (2008) has also given fuzzy nonlinear optimization. Torabi and Hassini (2008) described an interactive possibilistic programming approach for multiple objective supply chain master planning. Rehana and Mujumdar (2009) give an imprecise fuzzy risk approach for water quality management of a river system. Kanaya (2010) has discussed an interactive method for fuzzy multi objective nonlinear programming problem. Mesbah *et al.* (2010) have discussed the trading pollutant discharge permits in rivers using fuzzy nonlinear cost functions. Kheirfam (2010) has given sensitivity analysis for fuzzy linear programming problem with fuzzy variables. Ravi *et al.* (2010) have described a procedure for solving a fuzzy

nonlinear optimization problem by genetic algorithm. Li and Huang (2011) have discussed about the planning of agricultural water resources system associated with fuzzy and random features. Nikoo *et al.* (2013) have described optimal water and waste-load allocations in rivers using a fuzzy transformation technique. Xu and Qin (2013) have described a procedure for solving water quality management problem through combined genetic algorithm and fuzzy simulation.

In most agricultural planning problems, values of some parameters may not be known precisely. They are rather defined in a fuzzy sense. For successful handling of such problems, FGP techniques must be used. In the multi-criteria setting the special characteristic of GP models is the way the decision criteria is dealt with. Instead of the direct evaluation of the criteria, GP models explicitly introduce the desired target value for each criterion, and optimize the deviations of the criteria outcomes from these goals. The solution depends on the metrics used for the deviations and as well as the weighting method of the different goals. There are two common weighting methods where the first one is the fixed ordering of goals. In practice, this is implemented by searching a lexicographic minimum of the ordered deviation vector. The second one is the use of weights on goals and the minimization of the weighted sum of goal deviations. Sometimes, the minimization of the maximum deviation is also used as in Flawell (1976). The GP approach of multi-criteria problems has received increasing interest due to its modeling flexibility and conceptual simplicity. However, determining precisely the goal value of each objective is difficult for Decision Maker (DM) because possibly only partial information is known. To incorporate uncertainty and imprecision into the formulation, the fuzzy set theory developed by Zadeh (1965), helps in dealing with imprecision in real-world problems. This theory can also incorporate subjective characteristics in decision support systems. It is applied in several fields for its easy implementation, flexibility, tolerance of imprecise data and abilities in modeling nonlinear behavior of arbitrary complexity in terms of natural language. According to the fuzzy theory, the inaccurate objectives and constraints are represented by certain kind of membership functions, for instance, the triangle-like or trapezoid-like membership functions, we call the inaccurate objectives and constraints as fuzzy objectives and constraints. The concept of fuzzy programming (FP) in a general level was first proposed by Tanaka *et al.* (1974) under the framework of fuzzy decision of Bellman and Zadeh (1970). After

that, Zimmermann (1978) utilized FP approach to linear programming (LP) with several objectives. Introducing fuzzy uncertainty and imprecision into the goal programming problem, Narasimhan (1980) initially presented fuzzy goal programming (FGP) by using membership functions. It was further developed by Hannan (1981a), Hannan (1981b), Narasimhan (1981), Ignizio (1982), Tiwari *et al.* (1986 & 1987), Chen (1994)) and others. FGP has been used in agricultural planning by Slowinski (1986) and Sinha *et al.* (1988). Slowinski (1986) used FGP technique for a farm planning problem. Biswas and Pal (2005) applied FGP to a land use planning problem in an agricultural system in which utilization of total cultivable land, supply of productive resources, expected profit and expected production of various crops are defined fuzzily and some relative work was also done by Rao *et al.* (1988), Rao *et al.* (1992), Rubin and Narasimhan (1984). A brief introduction to fuzzy logic and some applications can be found in Hamalainen and Mantysaari (2002), Chen and Tasi (2001), Rasmy (2002) has described an expert system for multi-objective decision making as an application of fuzzy linguistic preferences and goal programming. Some of these applications are in the area of pattern recognition, data analysis, quality control, economy, operational research among others.

GP and FP are two approaches to solve the vector optimization problem by reducing it to scalar formulation. Both of them need an aspiration level for each goal. These aspiration levels are determined either by the DM or the decision analyst. In addition to the aspiration levels of the goals, FP needs admissible violation constants for each goal. The larger violation (or tolerance) of the goal indicates the lesser importance of this goal. It can be proved that every fuzzy linear program has an equivalent weighted linear goal program where the weights are the reciprocals of the admissible violation constants. In general, every FP is a GP with some weights assigned to the deviational variables in the objective function, where the FP has fuzziness in the aspiration level, i.e. to get a solution that makes the objectives as close as possible to a specific goal within a certain limit. In operations research, fuzzy set theory has been applied to techniques of linear and nonlinear programming, dynamic programming, queuing theory, multiple criteria decision-making, group decision-making.

## 1.8 STOCHASTIC PROGRAMMING

Stochastic programming deals with situations where some or all of the parameters of an MPP are random variables instead of deterministic quantities. A stochastic linear programming problem can be stated as:

$$\left. \begin{array}{l} \max \quad f(\underline{x}) = \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad \sum_{j=1}^n a_{ij} x_j \geq b_i; i = 1, 2, \dots, m \\ \text{and} \quad \underline{x} \geq 0, j = 1, 2, \dots, n \end{array} \right\} \quad (1.11)$$

where some or all of them  $c_j, a_{ij}$  and  $b_i$  are random variables with known probability distributions.

If all the functions (objective and/or constraints) are not linear the problem is termed as stochastic nonlinear programming.

## 1.9 CHANCE CONSTRAINED PROGRAMMING

The chance constrained programming problem was first studied by Charnes and Cooper (1959). The linear chance constrained programming problem may be expressed as

$$\left. \begin{array}{l} \min \quad f(\underline{x}) = \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad p \left[ \sum_{j=1}^n a_{ij} x_j \geq b_i \right] \geq p_i; i = 1, 2, \dots, m \\ \text{and} \quad \underline{x} \geq 0, j = 1, 2, \dots, n \end{array} \right\} \quad (1.12)$$

where all or some of the  $c_j, a_{ij}$  and  $b_i$  are random variables and  $p_i$  are specified probabilities close to 1. Inequalities in (1.12), that are,  $\sum_{j=1}^n a_{ij} x_j \geq b_i; i = 1, 2, \dots, m$  indicate that the  $i^{\text{th}}$  constraint has to be satisfied with a probability of at least  $p_i$ .

## **1.10 SAMPLE SURVEYS**

The term survey implies collecting informations either qualitative or quantitative on a finite set or subset of units constituting a population. For example, we may be interested in collecting information either on a set of persons or a set of animals or a set of plants or a set of households or a group of villages/cities or a group of business establishments or educational institutions, etc. The purpose of survey is to provide required information used for future planning or to assess its present status of the government departments, business concerns or research institutions. Surveys that cover the entire population under consideration are called “Census” or “Complete Enumeration”. On the other hand surveys that are based on a selected part of the population (called sample) are known as sample surveys. Surveys are varied in nature and may be conducted in variety of fields.

**1.10.1 Stratified Random Sampling:** The precision of an estimator of the population parameters depends on the size of the sample and the variability or heterogeneity among the units of the population. If the population is very heterogeneous and considerations of the cost limit the size of the sample, it may be found impossible to get a sufficiently precise estimate by taking a simple random sample from the entire population. The solution of this problem lies in stratified sampling design. In stratified sampling the size of population is divided into none overlapping and exhaustive groups called strata each of which is relatively more homogeneous as compared to the population as a whole. Independent simple random samples of predetermined sizes from each stratum are drawn and the required estimators of the population parameters are constructed.

### **Principal Reasons for Stratification:**

1. To gain in precision, we may divide a heterogeneous population into strata in such a way that each stratum is internally as homogeneous as possible.
2. For administrative convenience in organizing and supervising the field works, stratified sampling is best suited.
3. Administrative convenience may dictate the use of stratification; for example, the agency conducting the survey may have field offices, each of which can supervise the survey for a part of population.
4. To obtain separate estimates for some part of the population.

5. We can accommodate different sampling plans in different strata.
6. We can have data of known precision for certain sub divisions, consisting of one or more strata and each sub division is treated as a separate population.
7. Sampling problems may differ markedly in different parts of the population. With the human populations, people living in institutions like hotels and hospitals etc. are often placed in a different stratum from people living in ordinary homes because a different approach to the sampling is appropriate for the two situations.
8. Stratification may produce a gain in precision in the estimates of characteristics of the whole population. It may be possible to divide a heterogeneous population into subpopulations, each of which is internally homogeneous. This is suggested by the name strata, with its implication of a division into layers. If each stratum is homogeneous, in that the measurements vary little from one unit to another, a precise estimate of any stratum mean can be obtained from a small sample in that stratum. A precise estimate for the whole population can be obtained after combining these separate estimates.

In the theory of stratified sampling we come across with the study of the properties of estimates from a stratified sample and the best choice of the sample sizes to obtain maximum precision.

#### **1.10.2 Multivariate Stratified Sampling:**

In multivariate stratified sampling more than one (say  $p$ ) characteristics are defined on each unit of a stratified population.

The problem of optimum allocation in multivariate stratified sampling has drawn the attention of researchers for a long time starting apparently with Neyman (1934). It is felt that unless the strata variance for various characteristics are distributed in the same way, the classical Neyman allocation based on the variances of a single character is not of much use because an allocation which is optimum for one characteristics may not be acceptable for others. Due to this fact there is no unique or even widely accepted solution to the problem of optimum allocation in multivariate stratified sampling. One way to resolve this problem is to search for a compromise allocation, which is in some sense optimum for all the characteristics.



In multivariate surveys the problem of working out optimum sample sizes can be formulated as a multi-objective mathematical programming problem. The successful applications of mathematical programming techniques in the problems arising in univariate and multivariate sample surveys are due to the following authors:

Ghosh (1958), Aoyama (1963), Kokan (1963), Folk and Antle (1965), Ericson (1967), Kokan and Khan (1967), Kish (1967), Chatterjee (1966, 1967, 1968, 1972), Murthy (1967), Raj (1969), Chaddha *et al.* (1971), Ahsan (1975, 1978), Ahsan and Khan (1977, 1982), Cochran (1977), Omule (1985), Bethel (1989), McCallion (1992), Sheela and Unnithan (1992), Kreinbrock (1993), Rahim and Currie (1993), Jahan *et al.* (1994), Mandowara (1994), Jahan and Ahsan (1995), Csenki (1997), Khan *et al.* (1997), Bretthauer *et al.* (1999), Clark and Steel (2002), Ahsan *et al.* (2005), Varshney *et al.* (2011, 2012a), Diaz Garcia *et al.* (2007) have discussed the Optimum allocation in stratified surveys using stochastic programming. Diaz-Garcia *et al.* (2008) have described the multi-objective optimization for optimum allocation in multivariate stratified sampling. Dalenius and Gurney (1951), Dalenius and Hodge (1959), Ghosh (1963), Sethi (1963), Hartley (1965), Herlekar (1967), Serfling (1967), Buhler and Deutler (1975), Singh (1977), Unnithan (1978), Khan and Islam (1980), Bakhshi *et al.* (2010), Jarque (1981), Khare (1987), Miles *et al.* (1987), Miles and Robert (1989), Charnayak and Starytskyy (1998), Jahan *et al.* (2001), Khan *et al.* (2002), etc worked on optimum stratification. Dalenius (1957), Cochran (1963), Serfling (1968) and Khan *et al.* (1995) etc. worked on determining optimum number of strata. Kefitz (1951), Lahiri (1954), Murthy (1967), Raj (1969), Arthanari and Dodge (1981), Mitra and Pathak (1984), Aragon and Pathak (1990), Pathak and Fahimi (1992) etc worked on optimum integration of surveys.

Bethel (1989) expresses the optimal multi-character stratified sample allocation as a closed expression in terms of normalized lagrangian multipliers where as Rahim (1995) proposed an alternative procedure based on distance function of the sampling errors of all the estimates. Bosch and Wildner (2003) provided a generalization and unification of Neyman allocation for multivariate stratified populations. They demonstrated the usefulness of the proposed method through real world numerical examples. Kozak (2004) presented the formulae of the sample allocations in two schemes of the multivariate two-stage sampling. Kozak (2006) considered the problem of sample allocation between strata in multivariate surveys for a fixed

sample size. He discussed the problem under five different optimality criteria and their efficiencies are compared by performing a simulation study. Miller *et al.* (2007) presented optimal multivariate stratified sampling design that minimizes the weighted sum of relative variances for the estimation objectives. Cui and Zhu (2007) presented a straight forward approach to determine the sample size for estimating the mean vector of a multivariate population using Bonferroni inequality. Ansari *et al.* (2009) considered the problem of determining the fixed cost multiple response optimal sampling design by using the compromise criterion proposed by Chatterjee (1967). Khan *et al.* (2010) addressed the problem of optimum allocation in multivariate stratified sampling when auxiliary information is available. They worked out the separate and combined ratio and regression estimates of the population means through a stratified sample allocated by minimizing the weighted sum of the increases in the variance due to not using the individual optimum allocations subject to budgetary and other constraints. Ali *et al.* (2011) have given compromise allocation in multivariate stratified surveys with stochastic quadratic cost function. Ansari *et al.* (2011) have discussed an optimum multivariate multi-objective stratified sampling design and obtained an optimum allocation. Khan *et al.* (2012) have used allocation in multivariate stratified surveys with non-linear random cost function. Raghav *et al.* (2012) have obtained allocation of sample size in bi-objective stratified sampling using lexicographic goal programming. Iftekhhar *et al.* (2013) have obtained compromise solution in multivariate surveys with stochastic random cost function. Gupta *et al.* (2013) have developed an optimal chance constraint multivariate stratified sampling design using auxiliary information. Gupta *et al.* (2013) have used fuzzy goal programming approach in stochastic multivariate stratified sample surveys.

### **1.10.3 Double Sampling:**

In sample surveys, a number of sampling techniques like the use of ratio and regression estimates require information about an auxiliary variable  $x$  which is highly correlated with the main variable to increase the efficiency of the estimator of unknown parameters. There may be situations where such auxiliary information is not available but can be obtained relatively easily at a comparatively lower cost in terms of time and money. In such situations, it may be suitable to draw a relatively large preliminary sample and estimate the unknown auxiliary parameter and then take

either an independent sample, or a sub sample of the first sample for measuring the main variable of interest. This technique of estimating the auxiliary parameter first through a preliminary large sample and then drawing the second sample to measure and both is known as double sampling.

#### **1.10.4 Two-Stage Sampling:**

Suppose that each unit called the first stage unit (fsu) that are nearly homogeneous in the population can be divided into a number of smaller units ,or subunits called second stage units (ssu). A sample of fsu has been selected. If ssu within a selected fsu give similar results, it seems uneconomical to measure them all. A common practice is to select and measure a sample of the ssu from each choosen fsu. This technique is called sub sampling because the fsu is not measured completely but is itself sampled. Another name, due to Mahalanobis, is Two Stage Sampling, because the sample is taken in two stages. At first stage a sample of fsu often called the primary units is selected, then at the second stage a sample of ssu or sub units from each chosen fsu are selected for measurement. Two stages have a great variety of applications, which go far beyond the immediate scope of sample surveys. Whenever any process involves chemical, physical, or biological tests that can be performed on a small amount of material, it is likely to draw as a subsample from a larger amount that is itself a sample. The multivariate two-stage sampling design was discussed by Maqbool and Pirzada and an analytical solution.

#### **1.10.5 Three-Stage Sampling:**

The three stage sampling designs generally specifies three stages of selection: primary sampling units (PSUs) at the first stage, sub samples from PSUs at second stage as a secondary sampling units (SSUs) units and again sub samples from SSUs at third stage as a tertiary sampling units (TSUs).The three-stage sampling designs are well analyzed when one variable is measured. Different methods are available for obtaining the optimum allocation of sampling units to each stage. The problem of optimum allocation in three-stage sampling with one character is described in standard texts on sampling (Cochran (1977)). The estimation of optimal sample size for each characteristic is done one by one and then final sampling design can be selected among the individual solutions. For instance, in surveys to estimate crop production in India (Sukhatme, 1947), the village is a convenient sampling unit.

Within a village, only some of the fields growing the crop in question are selected so that the field is a sub-unit. When a field is selected, only certain parts of it are cut for the determination of yield per acre; thus the sub unit itself is sampled. Here we have to find the optimal sample sizes  $n$ ,  $m$  and  $p$  for all the three stages with the minimum cost.

#### **1.10.6 Non-Response in Sample Surveys**

Non-response is becoming a grooming concern in survey research. The phenomenon of non-response, when people are not able or willing to answer questions asked by the interviewer, can appear in sample surveys as well as in census. The extent and the effect of the non-response can vary greatly from one type of survey to another. It affects the quality of survey in two ways. Firstly, due to the reduction in the available amount of data, the estimates of population parameters will be less precise. Secondly, if a relationship exists between the variable under investigation and response behaviors, statements made on the basis of the response are not valid for the total population.

It is obvious that the extent of non-response must be kept as small as possible. In spite of many efforts, there still remains a considerable amount of non-response. Measures should be taken to prevent formation of wrong statements about the population. Combination of adjustment procedures and usual estimation techniques is necessary to yield valid population estimates.

In sample surveys, the population may be assumed to be composed of two parts:

- (i) Response group and
- (ii) Non-response group

In case when the units of the non-response group are such that after some additional efforts it is possible to get the information we refer such non-responding group of units as “Soft Core”. In some cases a part of non-response units are such that it is impossible to get information, the set of these units are referred as “Hard Core”. While estimating the population mean of the study character, the representation of responding and non responding units is required to give the representative value of the population mean. In case of “Soft Core”, the problem is to minimize the effect of non-response and make some adjustment which may provide the efficient estimate. In case of mail surveys, Hansen and Hurwitz (1946) have suggested the method of sub

sampling from the non-responding units of the sample and proposed the estimator for population mean with its standard errors. In case of stratified population, the problem of determining the initial sample size to be drawn and the value of sub sampling proportion for each stratum were considered by Khare (1987). Optimal sample designs in case of non-response have also been considered by El-Badry (1956), Ericson (1967) and Srinath (1971). Foradari (1961) generalized El-Badry's approach and also studied the uses of Hansen and Hurwitz's technique under different designs. Further improvement in the estimation of population mean in presence of non-response has been made by using information on auxiliary character. In this direction some conventional and alternate ratio, product and regression type estimators have been proposed when the population mean of auxiliary character is known or unknown (Rao (1986, 1987, 1990); Khare and Srivastva (1993, 2000); Tripathi and Khare (1997); Khare and Pandey (2000); Khare and Sinha (2002)), Khare and Sinha (2007). Fabian and Hyunshik (2000) proposed some ratio and regression estimators when population mean of auxiliary character is not known in advance. They used double sampling and also compared relative performances of the proposed estimators with the estimator by Hansen and Hurwitz (1946). Najmussehar and Bari (2002) have developed Double sampling for stratification with sub sampling the non-respondents. The problem of determining the optimum allocation to various strata and optimum size of subsamples among the non-respondents in multivariate stratified sampling is discussed by Khan *et al.* (2008), Raghav *et al.* (2010) have obtained an optimum sample sizes in case of stratified sampling for non-respondents. Raghav *et al.* (2012) have used multi-objective nonlinear programming approach in multivariate stratified sample surveys in the case of non-response. Raghav *et al.* (2012) have obtained an optimum sample sizes in case of stratified sampling for non-respondents; An integer solution. Haseen *et al.* (2012) have discussed a fuzzy approach for solving double sampling design in presence of non-response. Raghav *et al.* (2014) have discussed a new approach for obtaining allocation in multivariate stratified sampling in presence of non-response. Jeelani *et al.* (2014) have discussed the non-response problems in ranked set sampling.

#### **1.10.7 Randomized Response in Sample Surveys:**

The social survey generally contains sensitive topics such as habitual tax evasion, drunken driving, drug addiction, sexual behavior, family income etc. In such

situations, interviewees will be unwilling to give truthful answers. In order to reduce non-response, response bias and to promote respondent co-operation, improve upon the accuracy levels, a survey technique different from open or direct survey was needed that made people comfortable and encouraged to give truthful and faithful answers. Warner (1965) introduced a technique of generating 'randomized response' (RR) as a device to protect a respondent's privacy and secrecy for reducing the rate of non-response. The Warner's model requires the interviewee to give a "Yes" or "No" answer either to a sensitive question or to its negative, depending on the outcome of a randomizing device not disclosed to the interviewer. Several models for randomized response technique have been developed by Warner (1965), Kim and Flueck (1978), Franklin (1989), Singh and Singh (1992, 1993) and they use continuous randomized device instead of discrete one Mangat (1994) and Mangat *et al.* (1995) have proposed some other variates of Warner's model. Walt R. Simmons felt that the confidence of the respondents might be enhanced to the stigmatized characteristics. Following his suggestion, Hurwitz *et al.* (1967) developed a procedure and called it unrelated question randomized response model. In short he called it "U-model". Mishra and Sinha (1999) developed Warner's (1965) and Mangat and Singh's (1990, 1991) scheme to cover multiple type of statement and estimate the maximum likelihood estimates for the proportion along with its variance. Mangat *et al.* (1997) have also discussed the violation of respondent's privacy in Moor's model and its rectification through a random group strategy. Singh *et al.* (1999) considered the situation of multi-character surveys using randomized response technique in PPS sampling, where the study variable, besides being poorly correlated with the selection probabilities are also sensitive in nature. Chaudhuri (2001) has shown that Warner's (1965) randomized response (RR) model is applicable in complex surveys. Christofides (2003) has given an improved modification of Warner's (1965) pioneering randomized response (RR) technique in estimating an unknown proportion of people bearing a sensitive characteristic in a given community. Kim and Warde (2004) have described stratified Warner's randomized response model. Hussain *et al.* (2007) have described an alternative to Ryu, *et al.* randomized response model. Usman and Oshungade (2012) have given a two-way randomized response technique in stratification for tracking HIV Seroprevalence. Ghufuran *et al.* (2012, 2013) have formulated the two-stage stratified randomized response model as a mathematical programming problem and obtained the compromise allocations.

**Work done in this thesis:**

The research work presented in this thesis is spread over in six chapters.

**Chapter 1** is the introductory chapter. It provides an introduction to optimization, unconstrained optimization, constrained optimization, linear and non linear problems, integer programming and its applications in various fields, geometric programming and its applications in various emerging fields, multi-objective programming, bilevel programming and its uses, goal programming and its applications, fuzzy programming and fuzzy goal programming and their applications in different fields, stochastic programming, chance constrained programming, sampling and census, stratified random sampling, multivariate stratified sampling, double sampling, two and three stage sampling, non-response in sample surveys and randomized response in sample surveys.

Most of the numerical solutions to the illustrated examples in different chapters of this thesis are obtained by using the optimization software LINGO.

In **Chapter 2** we consider a multivariate stratified population with unknown strata weights. An optimum sampling design is proposed in the presence of non-response to estimate the unknown population means using double stratified sampling (DSS) strategy. The problem is formulated as a multi-objective integer nonlinear programming problem (MOINLPP). The problem turns out to be a non-linear bilevel programming problem. Then a fuzzy goal programming approach is used to solve the non-linear bilevel programming problem. The objective function at each level is non-linear in nature and there is one linear constraint with some upper and lower bounds. A compromise optimum allocation has been obtained in the minimum number of steps. The work of this chapter is based on my research paper published in "International Journal of Scientific & Engineering Research" (FRANCE).

**Chapter 3** provides the description of a multivariate stratified sampling problem with non linear objective function and a probabilistic non linear cost constraint. The problem is formulated as a multi-objective non-linear programming problem (MONLPP). The fuzzy goal programming (FGP) approach has been used to solve the stochastic multivariate stratified sampling problem with non-linear objective function and probabilistic non-linear cost constraint. In the model formulation of the problem, we first determine the individual best solution of the objective functions subject to the

system constraints and construct the non-linear membership functions of each objective. The non linear membership functions are then transformed into equivalent linear membership functions by first order Taylor series at the individual best solution point. Fuzzy goal programming approach is then used to achieve maximum degree of each of the membership goals by minimizing negative deviational variables and finally obtain the compromise allocation. The work of this chapter is based on my research paper published in “The South Pacific Journal of Natural and Applied Sciences” (CISRO, FIJI).

In **Chapter 4** an optimum allocation in three-stage sampling design with three variables is considered. Presently, the solution of mathematical programming problems, sampling problems, engineering problems and management problems etc. are very much dependent upon the efficiency of the software. With this regard, we have provided an effective method being handled manually and also by using LINGO software for obtaining optimum allocations in three-stage sampling with the help of primal-dual relationship of GP. The manual description of the solution procedure of GP is very simple in comparison to the complex analytical techniques used in statistical literature. There may not be precise knowledge of parameters in the GP in real world due to insufficient information. The feasibility and effectiveness of the present approach has been illustrated by numerical example. GP optimization technique can be utilized in double sampling design having multiple characters; this is the wider application of the proposed approach. The work of this chapter is based on my research paper published in “International Journal of Scientific & Engineering Research” (FRANCE).

**Chapter 5** provides the use of fuzzy programming for solving a multi-objective geometric programming problem (MOGPP). The problem of non-response with significant travel costs where the cost is quadratic in  $\sqrt{n_h}$  in multivariate stratified sample surveys is formulated as an MOGPP. The fuzzy programming approach is described for solving the formulated MOGPP. The formulated MOGPP is solved with the help of LINGO software and the dual solution is obtained. The optimum allocation of sample sizes of respondents and non-respondents are obtained with the help of dual solution and primal-dual relationship theorem. An illustrative numerical



example is given to ascertain the practical utility of the proposed method in sample survey problems in the presence of non-response. The work of this chapter is based on my research paper published in “American Journal of Operations Research”.

In **Chapter 6** the two-stage randomized response (*RR*) model in multivariate stratified sample surveys is considered. The problem is formulated as a multi-objective nonlinear programming problem. A complete method of solution of the formulated problem is projected to solve the problem. A numerical example is worked out to illustrate the computational details of the proposed method. In the next part of this chapter the two-stage stratified Warner’s randomized response (*RR*) model with travel cost is considered and fuzzy geometric programming approach is used to obtain the optimum allocations of sample sizes. The chances of non-response in a multivariate stratified sample survey when the sampling is done with the sensitive questions may be significantly high. To ascertain the practical utility of the fuzzy geometric programming approach in sample surveys problem of randomized response model is extended for multiple sensitive questions to illustrate the fuzzy geometric programming procedure. The work of this chapter is based on my research paper communicated in “Journal of Mathematical Modeling and Algorithms in Operations Research”.

## **CHAPTER – II**

### **Bi-level Fuzzy Goal Programming Approach in Double Sampling Design in Presence of Non-Response**

**CHAPTER – II**  
**BILEVEL FUZZY GOAL PROGRAMMING APPROACH IN**  
**DOUBLE SAMPLING DESIGN IN PRESENCE OF NON-RESPONSE**

**2.1 Introduction**

In sample surveys we often experience the problem of non-response. Non-response means that the desired information is not obtained for all units selected in the sample for one reason or the other. For example, if the sampling unit is an individual then the selected person may not be willing to provide the required information or he may not be at home when the interviewer called. In case of non-response the sampler has an incomplete sample data that affects the quality of estimates of the unknown population parameters. Hansen and Hurwitz (1946) were first who dealt with the problem of non-response in mail surveys. They selected a preliminary sample and mailed the questionnaires to all the selected units. Non-respondents are identified and a second attempt was made by interviewing a subsample of non-respondents. They constructed the estimate of the population mean by combining the data from the two attempts and derived the expression for the sampling variance of the estimate. The optimum sampling fraction among the non-respondents is also obtained. El-Badry (1956) extended the Hansen and Hurwitz's technique by sending waves of questionnaire to the non-respondents units to increase the response rate. Khare (1987) investigated the problem of optimum allocation in stratified sampling in presence of non-response for fixed cost as well as for fixed precision estimate.

The problem of optimum allocation in stratified random sampling is well known in sampling literature for a univariate population. Work is done in this respect by Cochran (1977) and Sukhatme *et al.* (1984). But when more than one characteristic are under study then it is not possible to use individual optimum allocation to each strata because allocation which is optimum for one characteristic may not be optimum for the other characteristic. There should be a positive strong correlation between the characteristics under study. Thus; usually one has to use an optimum allocation that is optimum in 'some sense' for all the characteristics. Such an allocation is known as a compromise allocation in sampling literature. Methods for solving the problem of optimum allocation in multivariate stratified sampling are given by Geary (1949), Dalenius (1957), Ghosh (1958), Yates (1960), Aoyama (1963), Folks and Antle (1965), Chatterjee (1967, 1968), Kokan and Khan (1967), Ahsan (1975-1976, 1978),

Ahsan and Khan (1977), Bethel (1985, 1989), Schittkowski (1985-1986), Chromy (1987), Jahan *et al.* (1994, 1995), Khan *et al.* (1997), Khan *et al.* (2003, 2008), Singh (2003), Diazandom-Gracia and cortez (2006, 2008), Kozak (2006, 2006) and many others suggested new compromise criterion further the existing criteria under various situations.

When some auxiliary information is available, it may be used to increase the precision of the estimate. Ige and Tripathi (1987), Rao (1973), Tripathi and Bhal (1991) and some other authors discussed the use of auxiliary information in stratified sampling using double sampling technique.

The problems of optimum allocation, where the strata weights are unknown and non-response also occurs have been studied by some authors. Okafor (1994) solved the above problem for stratified population in univariate case using a double sampling strategy (DDS). The same problem was also formulated by Najmussehar and Bari (2002) using dynamic programming technique to obtain a solution. A comparative study has also been done by Varshney *et al.* (2011) by developing a goal programming technique to solve the problem.

Bilevel programming problems form an important class of optimization problems involving hierarchical decision making processes where the upper level decision maker (ULDM) anticipates the responses from the lower level decision maker (LLDM) and proceeds with optimizing its own objective. Their origin traces back to the Stackelberg competition models in economics. The ULDM is called the leader's problem and that the LLDM is called the follower's problem. The follower executes its policies after and in view of the decisions of the upper level decision maker. Control over the decision variables is partitioned among the levels but a decision variable of one level may affect the objective function of other level.

In this chapter, we have considered the problem of determining a compromise allocation in multivariate stratified random sampling with the unknown strata weights and in the presence of non-response. The strata weights are estimated using double sampling. The problem of obtaining a compromise allocation has been formulated as a non-linear bi-level programming problem. Fuzzy goal programming (FGP) approach is used to work out the compromise allocation of the non-linear bi-level programming problem in which we define the membership functions of each objective function and then transformed the membership functions into equivalent linear membership functions with the help of first order Taylor series and finally by

forming the fuzzy goal programming model, the desired compromise allocations with integer values are obtained directly by the optimization software LINGO. Also a numerical example has been presented to illustrate the computational details.

## 2.2 Double sampling for stratification in presence of non-response

Let a multivariate survey be designed to estimate the number of persons suffering from certain specific diseases like Diabetes, High Blood Pressure, Cataract, Glaucoma, HIV etc. in a city having a population of size  $N$ , divided into three strata according to the family income. The information is to be obtained through mailed questionnaires. Further let the actual sizes of the strata say  $N_1, N_2$  and  $N_3$  be not known. In mailed questionnaire surveys usually the problem of non-response is also present. Under the above circumstances the surveyor may use the technique discussed in this manuscript.

Consider a population of size  $N$ , divided into  $L$  non-overlapping strata of sizes  $N_1, N_2, \dots, N_L$  where  $\sum_{h=1}^L N_h = N$ . If  $N_1, N_2, \dots, N_L$  are not known in advance then the strata weights  $W_h = N_h / N; h=1, 2, \dots, L$  remain unknown. In such a situation double sampling technique may be used to estimate the unknown  $W_h$  by taking a large preliminary sample of size  $n'$ , treating the population as unstratified. The units  $n'_h; h=1, 2, \dots, L$  of the sample falling in each stratum are recorded. An unbiased of  $W_h$  is given by  $w_h = n'_h / n'$ . Subsample of  $n_h = v_h n'_h; h=1, 2, \dots, L; 0 \leq v_h \leq 1$  is then drawn out of  $n'_h$  units using srsWOR from each stratum for fixed  $v_h$ . The double sampling for stratification (DSS) estimator of the population mean  $\bar{Y}_j$  of the  $j^{th}$  characteristic out of  $p$  characteristics measured on each selected unit is given as:

$$\bar{y}_{jds} = \sum_{h=1}^L w_h \bar{y}_{jh} \quad (2.1)$$

where  $\bar{y}_{jh} = 1/n_h \sum_{i=1}^{n_h} y_{jhi}$  is the sample mean of  $j^{th}$  characteristic,  $j=1, 2, \dots, p$  based on  $n_h$  units for stratum  $h$  and 'ds' stands for double sampling. The sampling variance of  $\bar{y}_{jds}$  is given as:

$$V(\bar{y}_{jds}) = \left( \frac{1}{n'} - \frac{1}{N} \right) S_j^2 + \frac{1}{n'} \sum_{h=1}^L w_h \left( \frac{1}{v_h} - 1 \right) S_{jh}^2 \quad (2.2)$$

$$C = c_0 n' + \sum_{h=1}^L c_{h1} n_h + \sum_{h=1}^L c_{h11} n_{h1} + \sum_{h=1}^L c_{h12} m_{h2} \quad (2.5)$$

where  $c_0$  is per unit cost of getting information from the preliminary sample,  $c_{h1}$  is per unit cost of making the first attempt (phase-I),

$c_{h11} = \sum_{j=1}^p c_{jh11}$  is the per unit cost for processing the result of all the  $p$  characteristics on the  $n_{h1}$  selected units from respondents group in the  $h^{th}$  stratum at phase-I,

$c_{h12} = \sum_{j=1}^p c_{jh12}$  is the per unit cost for measuring and processing the results of all the  $p$  characteristics on the  $m_{h2}$  units selected from the non-respondents group in the  $h^{th}$  stratum at the second attempt (phase-II),

$c_{jh11}$  and  $c_{jh12}$  are the per unit costs of measuring the  $j^{th}$  characteristics at phase-I and phase-II respectively.

Since  $n_{h1}$  is not known until the first attempt has been made, the quantity  $w_{h1} n_h$  may be used as its estimated value. The total expected cost  $\hat{C}$  of the survey is thus given as:

$$\hat{C} = c_0 n' + \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h + \sum_{h=1}^L c_{h12} m_{h2} \quad (2.6)$$

### 2.3 Formulation of the problem

Now if we encounter a multivariate problem where it is given that a certain character must be given priority over other characters and has control over particular sample sizes then that problem can be solved using bilevel programming problem (BPP). Let us consider a multivariate problem partitioned into four strata with two characters where variance of one character is given priority over other and controls the certain strata size. In this way the multivariate problem can be solved as a bilevel programming problem.

Now the formulation of the problem for phase-I where the problem is to find the optimum sizes of the subsamples  $n_h; h = 1, 2, \dots, L$  which may be obtained by minimizing  $V_j; j = 1, 2$  for the fixed cost or by minimizing the cost for fixed variance may be given as:

$$\left. \begin{array}{l}
\min_{n_1, n_2} V_1 \text{ where } n_3, n_4 \text{ solves} \\
\min_{n_3, n_4} V_2 \\
\text{subject to } \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h + \sum_{h=1}^L c_{h12} m_{h2} \leq (\hat{C} - c_0 n') \\
2 \leq n_h \leq n'_h \\
\text{and } n_h \text{ integers; } h = 1, 2, \dots, L
\end{array} \right\} \quad (2.7)$$

where  $V_j; j = 1, 2$  are as defined in (2.4).

Ignoring the terms independent of  $n_h$  minimizing  $V_j$  will be equivalent to minimize

$$\begin{aligned}
Z_j(n_1, n_2, \dots, n_L) &= \frac{1}{n'} \sum_{h=1}^L \left( \frac{w_h n'_h S_{jh}^2 + w_h \left( (1 - k_h^*) / k_h^* \right) n'_h S_{jh2}^2}{n_h} \right) \\
&= \sum_{h=1}^L \frac{a_{jh}}{n_h} \\
&= Z_j; j = 1, 2
\end{aligned} \quad (2.8)$$

$$\text{where } a_{jh} = \left( \frac{w_h n'_h S_{jh}^2 + w_h \left( (1 - k_h^*) / k_h^* \right) n'_h S_{jh2}^2}{n'} \right)$$

The cost constraint may be expressed as:

$$\sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h \leq \hat{C}_0; \text{ where } \hat{C}_0 = \hat{C} - c_0 n' - \sum_{h=1}^L c_{h12} m_{h2}$$

Problem (2.7) may be restated as:

$$\left. \begin{array}{l}
\min_{n_1, n_2} Z_1 \text{ where } n_3, n_4 \text{ solves} \quad (\text{upper level}) \\
\min_{n_3, n_4} Z_2 \quad (\text{lower level}) \\
\text{subject to } \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h \leq \hat{C}_0 \\
2 \leq n_h \leq n'_h \\
\text{and } n_h \text{ integers; } h = 1, 2, \dots, L
\end{array} \right\} \quad (2.9)$$

where  $Z_j; j = 1, 2$  are as defined in (2.8).

At phase-II, the problem is to work out the optimum values of  $m_{h2}$  which may be obtained by minimizing  $V_j; j = 1, 2$  given by (2.4) for given cost in (2.6).

Ignoring the terms independent of  $m_{h2}$  in the RHS of (2.4), substituting

$k_h^* = m_{h2} / n_{h2}$  and  $v_h = n_{h1} / n_h$ , and the problem (2.9) may be stated as:

where  $S_j^2 = \frac{1}{N-1} \sum_{i=1}^N (y_{ji} - \bar{Y}_j)^2$  is the population of  $j^{th}$  characteristic based on  $N$  units and  $S_{jh}^2 = \frac{1}{N_h-1} \sum_{i=1}^{N_h} (y_{jhi} - \bar{Y}_{jh})^2$  is the population variance for  $j^{th}$  characteristic based on  $N_h$  units for stratum  $h$ . The expression (2.1) and (2.2) assume total response.

In the presence of non-response, let  $n_{h1}$  units respond at the first call and  $n_{h2}$  units denote the number of non-respondents out of  $n_h$  units. Using Hansen and Hurwitz technique, a subsample of non-respondents of size  $m_{h2} = k_h^* n_{h2}; 0 < k_h^* < 1$  out of  $n_{h2}$  units is drawn and interviewed with improved method where  $k_h^*$  is a known constant.

For  $j^{th}$  characteristic, an unbiased estimator  $\bar{y}_{jds}^*$  for  $\bar{Y}_j$  based on sample means from the respondents and the non-respondents group obtained in second attempt is given as:

$$\bar{y}_{jds}^* = \sum_{h=1}^L w_h \bar{y}_{jh}^* \quad (2.3)$$

$$\text{where } \bar{y}_{jh}^* = \frac{n_{h1} \bar{y}_{jh1} + n_{h2} \bar{y}_{jm_{h2}}}{n_h}$$

$\bar{y}_{jh1}$  sample mean for respondents based on  $n_{h1}$  units

$\bar{y}_{jm_{h2}}$  sample mean for the non-respondents based on  $m_{h2}$  units (second attempt)

The variance of  $\bar{y}_{jds}^*$  is given as:

$$\begin{aligned} V(\bar{y}_{jds}^*) &= V(\bar{y}_{jds}) + \frac{1}{n'} \sum_{h=1}^L w_{h2} \left( \frac{1-k_h^*}{k_h^* v_h} \right) S_{jh2}^2 \\ &= \left( \frac{1}{n'} - \frac{1}{N} \right) S_j^2 + \frac{1}{n'} \sum_{h=1}^L w_h \left( \frac{1}{v_h} - 1 \right) S_{jh}^2 + \frac{1}{n'} \sum_{h=1}^L w_{h2} \left( \frac{1-k_h^*}{k_h^* v_h} \right) S_{jh2}^2 \\ &= V_j; \quad j = 1, 2, \dots, p \end{aligned} \quad (2.4)$$

where  $w_{h2} = n_{h2} / n_h$  is the proportion of the non-respondents and  $S_{jh2}^2$  is the population variance of  $j^{th}$  characteristic,  $j = 1, 2, \dots, p$  of the non-respondents in  $h^{th}$  stratum.

Assuming a linear cost function the total cost of the survey may be given as:



$$\left. \begin{aligned}
& \min_{m_{12}, m_{22}} Z_1' \text{ where } m_{32}, m_{42} \text{ solves (upper level)} \\
& \min_{m_{32}, m_{42}} Z_2' \text{ (lower level)} \\
& \text{subject to } \sum_{h=1}^L c_{h12} m_{h2} \leq \hat{C}_0' \\
& \quad 2 \leq m_{h2} \leq n_{h2} \\
& \text{and } m_{h2} \text{ integers, } h = 1, 2, \dots, L
\end{aligned} \right\} \quad (2.10)$$

where  $Z_j'; j = 1, 2$  are the function of  $m_{h2}; h = 1, 2, \dots, L$  given by

$$\begin{aligned}
Z_j'(m_{12}, m_{22}, \dots, m_{L2}) &= \frac{1}{n'} \sum_{h=1}^L \frac{w_{h2} n_{h2} n_h' S_{jh2}^2}{m_{h2} n_h} \\
&= \sum_{h=1}^L \frac{b_{jh}}{m_{h2}}; j = 1, 2
\end{aligned}$$

$$b_{jh} = \frac{w_{h2} n_{h2} n_h' S_{jh2}^2}{n' n_h} \text{ and } \hat{C}_0' = \hat{C} - c_0 n' - \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h.$$

#### 2.4 The solution procedure by using fuzzy goal programming

We now formulate the fuzzy programming model of NLBLPP by transforming the objective functions  $Z_1$  and  $Z_2$  into fuzzy goals by means of assigning an imprecise aspiration level to each of them. Let  $Z_1^*$  and  $Z_2^*$  be the optimal solutions of the objective functions of ULDM and LLDM respectively, when calculated in isolation subject to the system constraints.

Then the fuzzy goals appear in the form:

$$Z_1 \gtrsim Z_1^* \text{ and } Z_2 \gtrsim Z_2^*$$

Using the individual best solutions, we formulate a payoff matrix as follows:

$$\begin{array}{cc}
Z_1(\bar{n}) & Z_2(\bar{n}) \\
\begin{array}{l} (n^{(1)}) \\ (n^{(2)}) \end{array} & \left[ \begin{array}{cc} Z_1(n_h^{(1)}) & Z_2(n_h^{(1)}) \\ Z_1(n_h^{(2)}) & Z_2(n_h^{(2)}) \end{array} \right], \text{ where } h = 1, 2, \dots, L
\end{array}$$

where  $n_h^{(1)}$  &  $n_h^{(2)}$  are the individual optimal points of the objective functions of ULDM and LLDM.

The maximum value of each column gives the upper tolerance limit for the objective functions  $Z_1$  and  $Z_2$ . The minimum value of each column gives lower tolerance limit for the objective functions.

The objective value, which is equal to or larger than  $Z_1^*$  should be absolutely satisfactory to ULDM. Similarly, the objective values, which is equal to or larger than  $Z_2^*$  should be absolutely satisfactory to LLDM. If the individual best solutions are identical, then a satisfactory optimal solution of the system is reached. However, this situation arises rarely because the objective of ULDM and LLDM is conflicting in general.

The non-linear membership function  $\mu_1(\bar{n})$  corresponding to the objective function  $Z_1(\bar{n})$  of the ULDM can be formulated as:

$$\mu_1(\bar{n}) = \begin{cases} 0, & \text{if } Z_1(\bar{n}) \geq Z_1^u(\bar{n}) \\ \frac{Z_1^u(\bar{n}) - Z_1(\bar{n})}{Z_1^u(\bar{n}) - Z_1^L(\bar{n})}, & \text{if } Z_1^L(\bar{n}) \leq Z_1(\bar{n}) \leq Z_1^u(\bar{n}) \\ 1, & \text{if } Z_1^u(\bar{n}) \leq Z_1^L(\bar{n}) \end{cases}$$

Here  $Z_1^u(\bar{n})$  and  $Z_1^L(\bar{n})$  are respectively the upper and lower tolerance limits of the fuzzy objective goal for ULDM.

Similarly, the non-linear membership function  $\mu_2(\bar{n})$  corresponding to the objective function  $Z_2(\bar{n})$  of the LLDM can be formulated as:

$$\mu_2(\bar{n}) = \begin{cases} 0, & \text{if } Z_2(\bar{n}) \geq Z_2^u(\bar{n}) \\ \frac{Z_2^u(\bar{n}) - Z_2(\bar{n})}{Z_2^u(\bar{n}) - Z_2^L(\bar{n})}, & \text{if } Z_2^L(\bar{n}) \leq Z_2(\bar{n}) \leq Z_2^u(\bar{n}) \\ 1, & \text{if } Z_2^u(\bar{n}) \leq Z_2^L(\bar{n}) \end{cases}$$

Here  $Z_2^u(\bar{n})$  and  $Z_2^L(\bar{n})$  are respectively the upper and lower tolerance limits of the fuzzy objective goal for LLDM. Now the problem of phase-I reduces to

$$\left. \begin{array}{l} \min \mu_1(\bar{n}) \\ \min \mu_2(\bar{n}) \\ \text{subject to } \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h \leq \hat{C}_0 \\ \quad \quad \quad 2 \leq n_h \leq n_h^* \\ \text{and } \quad \quad n_h \text{ integers; } h=1,2,\dots,L \end{array} \right\} \quad (2.11)$$

### 2.4.1 Linearization of the non-linear membership functions by first order Taylor Series

Let  $\bar{n}_h^{(1)*}$  and  $\bar{n}_h^{(2)*}$  be the individual best solutions of the non-linear membership functions  $\mu_1(\bar{n})$  and  $\mu_2(\bar{n})$  subject to the constraints. Now, we transform the non-linear membership functions  $\mu_1(\bar{n})$  and  $\mu_2(\bar{n})$  into equivalent linear membership functions at individual best solution point by first order Taylor series as follows:

$$\begin{aligned}\mu_1(\bar{n}) &\cong \mu_1(\bar{n}_h^{(1)*}) + (n_1 - n_1^{(1)*}) \frac{\partial}{\partial n_1} \mu_1(\bar{n}_h^{(1)*}) + \dots + (n_L - n_L^{(1)*}) \frac{\partial}{\partial n_L} \mu_1(\bar{n}_h^{(1)*}) = \xi_1(\bar{n}) \\ \mu_2(\bar{n}) &\cong \mu_2(\bar{n}_h^{(2)*}) + (n_1 - n_1^{(2)*}) \frac{\partial}{\partial n_1} \mu_2(\bar{n}_h^{(2)*}) + \dots + (n_L - n_L^{(2)*}) \frac{\partial}{\partial n_L} \mu_2(\bar{n}_h^{(2)*}) = \xi_2(\bar{n})\end{aligned}$$

### 2.4.2 FGP model of NLBLPP

The NLBLPP represented by (2.11) reduces to the following problem

$$\left. \begin{array}{l} \min \xi_1(\bar{n}) \\ \min \xi_2(\bar{n}) \\ \text{subject to } \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h \leq \hat{C}_0 \\ \quad \quad \quad 2 \leq n_h \leq n_h' \\ \text{and } \quad \quad n_h \text{ integers; } h = 1, 2, \dots, L \end{array} \right\} \quad (2.12)$$

The maximum value of a membership function is unity (one), so for the defined membership functions in (2.12), the flexible membership goals having the aspiration level unity can be presented as:

$$\begin{aligned}\xi_1(\bar{n}) + \delta_1 &= 1 \\ \xi_2(\bar{n}) + \delta_2 &= 1\end{aligned}$$

Here  $\delta_1 \geq 0, \delta_2 \geq 0$  are the over deviational variables.

Then our fuzzy goal programming model for phase-I is:

$$\left. \begin{array}{l}
\min \delta_1 + \delta_2 \\
\text{subject to } \xi_1(\bar{n}) + \delta_1 = 1 \\
\quad \quad \quad \xi_2(\bar{n}) + \delta_2 = 1 \\
\quad \quad \quad \sum_{h=1}^L (c_{h1} + w_{h1} c_{h11}) n_h \leq \hat{C}_0 \\
\quad \quad \quad 2 \leq n_h \leq n_h' \\
\quad \quad \quad \delta_1 \geq 0 \\
\quad \quad \quad \delta_2 \geq 0 \\
\text{and } n_h \text{ integers; } h = 1, 2, \dots, L
\end{array} \right\} \quad (2.13)$$

Similarly we can formulate the fuzzy goal programming model for phase-II.

## 2.5 FGP Algorithm for NLBLPP

From the discussion of the previous section, the FGP algorithm for solving NLBLPP can be outlined as given below:

**Step 1:** Find the individual best solution of objective function for the levels subject to the system constraints.

**Step 2:** Formulate the payoff matrix. Then define upper and lower tolerance limits of each objective function.

**Step 3:** Construct non-linear membership function  $\mu_1(\bar{n})$  corresponding to objective function  $Z_1(\bar{n})$  of ULDM. Similarly, construct non-linear membership function  $\mu_2(\bar{n})$  corresponding to the objective function  $Z_2(\bar{n})$  of LLDM.

**Step 4:** Find the individual best solution of the non-linear membership functions  $\mu_1(\bar{n})$  &  $\mu_2(\bar{n})$  subject to the system constraints.

**Step 5:** Transform the non-linear membership functions  $\mu_1(\bar{n})$  &  $\mu_2(\bar{n})$  into equivalent linear membership functions  $\xi_1(\bar{n})$  &  $\xi_2(\bar{n})$  respectively at the individual best solution point by first order Taylor series.

**Step 6:** Formulate the FGP model for NLBLPP.

**Step 7:** Solve the FGP model using LINGO software.

**Step 8:** End.

## 2.6 Numerical Illustration:

For the purpose of demonstrating the use of DSS, the following numerical data has been taken from Khan et al. (2008). It illustrates the proposed technique for computing the values of overall optimum allocations and the optimum sample sizes from non-respondents at phase-II. A population of size  $N = 3850$  is divided into four strata. Two characteristics are defined on each unit of the population. It is assumed that the estimation of population means of the two characteristics is of interest.

**Table 1: Data for four Strata and two characteristics:**

$h$	$w_h$	$S_{1h}^2$	$S_{2h}^2$	$v_h$	$k_h^*$	$c_{h1}$	$c_{h11}$	$c_{h12}$
1	0.32	4817.72	8121.15	0.4	0.5	1	2	3
2	0.21	6251.26	7613.52	0.5	0.6	1	3	4
3	0.27	3066.16	1456.4	0.6	0.7	1	4	5
4	0.20	6207.25	6977.72	0.65	0.75	1	5	6

**Table 2: Subdivided data as respondent and non respondent groups for four strata two with characteristics:**

$h$	Groups	$S_{1h}^2$	$S_{2h}^2$	$w_{hk}$ $K=1,2$
1	Respondent	2218.74	4318.28	$w_{11} = 0.70$
	Non-respondent	1908.37	2557.62	$w_{12} = 0.30$
2	Respondent	4056.75	5067.26	$w_{21} = 0.80$
	Non-respondent	3541.23	3984.85	$w_{22} = 0.20$
3	Respondent	2785.15	957.56	$w_{31} = 0.75$
	Non-respondent	1677.65	877.13	$w_{32} = 0.70$
4	Respondent	5015.17	3085.78	$w_{41} = 0.72$
	Non-respondent	2156.52	2756.62	$w_{42} = 0.28$

Table 1 shows the available information. Each stratum is further subdivided into respondent and non-respondent groups as given in Table 2. It is assumed that  $v_h$  and  $k_h^*$  are known and the preliminary sample size  $n'=1000$ , in the last column of the Table  $2k=1$  for respondent group and  $k=2$  for non-respondent group. Further let the total amount available for the survey be  $C=3,000$  units. Out of these 3,000 units 750 units are earmarked for the preliminary sample of size  $n'$ , 1,900 units are remarked for phase-I and 350 units are remarked for phase-II. Using estimated values of strata weights and the size of selected preliminary sample, the values of  $n'_h = w_h n'$ ;  $h=1,2,\dots,L$  are obtained as:

After substituting the values from tables 1 and 2, the NLBLPP (2.9) for the first phase becomes

$$\left. \begin{aligned} \min Z_1 &= \frac{676.53805}{n_1} + \frac{374.83501}{n_2} + \frac{272.05508}{n_3} + \frac{288.54504}{n_4} \\ \min Z_2 &= \frac{1077.13728}{n_1} + \frac{447.33203}{n_2} + \frac{131.54568}{n_3} + \frac{330.56571}{n_4} \\ \text{subject to } &2.4n_1 + 3.4n_2 + 4n_3 + 4.6n_4 \leq 1900 \\ &2 \leq n_1 \leq 320 \\ &2 \leq n_2 \leq 210 \\ &2 \leq n_3 \leq 270 \\ &2 \leq n_4 \leq 200 \\ \text{and } &n_h \text{ integers; } h=1,2,\dots,4 \end{aligned} \right\}$$

$Z_1^* = 11.12985$  at  $(219, 139, 107, 103)$  and  $Z_2^* = 12.12468$  at  $(264, 143, 72, 107)$  are the individual best solutions of both the levels which is provided by software LINGO.

Then the fuzzy goals appear as:

$$Z_1(\tilde{n}) \gtrsim 11.12985 \text{ and } Z_2(\tilde{n}) \gtrsim 12.12468$$

Now the pay-off matrix of the above problem is given below:

$$\begin{array}{cc} & \begin{matrix} Z_1(n) & Z_2(n) \end{matrix} \\ \begin{pmatrix} n^{(1)} \\ n^{(2)} \end{pmatrix} & \begin{bmatrix} 11.12985 & 12.57542 \\ 11.65909 & 12.12468 \end{bmatrix} \end{array}$$

Here  $Z_1''(\bar{n}) = 11.65909$ ,  $Z_1''(\bar{n}) = 11.12985$  and  $Z_2''(\bar{n}) = 12.57542$ ,  $Z_2''(\bar{n}) = 12.12468$  are the upper and lower tolerance limits.

The non-linear membership functions of ULDM and LLDM are:

$$\mu_1(\bar{n}) = \frac{11.65909 - Z_1(\bar{n})}{11.65909 - 11.12985}, \quad \mu_2(\bar{n}) = \frac{12.57542 - Z_2(\bar{n})}{12.12468 - 3.16025}$$

The membership function  $\mu_1(\bar{n})$  is minimal at the point (219,139,107,103) and membership function  $\mu_2(\bar{n})$  is minimal at the point (264,143,72,107) respectively.

Then, the non-linear membership functions are transformed into linear at the individual best solution point by first order Taylor polynomial series as follows:

$$\begin{aligned} \mu_1(\bar{n}) &\cong 1 + (n_1 - 219) \times 0.0267 + (n_2 - 139) \times 0.0367 \\ &\quad + (n_3 - 107) \times 0.0449 + (n_4 - 103) \times 0.0514 = \xi_1(\bar{n}) \\ \mu_2(\bar{n}) &\cong 1 + (n_1 - 264) \times 0.0343 + (n_2 - 143) \times 0.0485 \\ &\quad + (n_3 - 72) \times 0.0563 + (n_4 - 107) \times 0.0641 = \xi_2(\bar{n}) \end{aligned}$$

Then, the FGP model for solving NLBLPP is formulated as follows:

$$\left. \begin{aligned} \min \quad & \delta_1 + \delta_2 \\ \text{subject to} \quad & 1 + (n_1 - 219) \times 0.0267 + (n_2 - 139) \times 0.0367 + \\ & (n_3 - 107) \times 0.0449 + (n_4 - 103) \times 0.0514 + \delta_1 = 1 \\ & 1 + (n_1 - 264) \times 0.0343 + (n_2 - 143) \times 0.0485 + \\ & (n_3 - 72) \times 0.0563 + (n_4 - 107) \times 0.0641 + \delta_2 = 1 \\ & 2.4n_1 + 3.4n_2 + 4n_3 + 4.6n_4 \leq 1900 \\ & 2 \leq n_1 \leq 320 \\ & 2 \leq n_2 \leq 210 \\ & 2 \leq n_3 \leq 270 \\ & 2 \leq n_4 \leq 200 \\ & \delta_1 \geq 0 \\ & \delta_2 \geq 0 \end{aligned} \right\}$$

and  $n_h$  integers;  $h = 1, 2, \dots, 4$

By solving the FGP model by software LINGO, we get the optimal solution as:

$$n_1^* = 279, n_2^* = 107, n_3^* = 70, n_4^* = 127 \text{ with } Z_1^* = 12.08651 \text{ and } Z_2^* = 12.52348.$$

Similarly we formulate the FGP model for phase-II as given below:

$$\begin{array}{l}
\min \delta_1 + \delta_2 \\
\text{subject to } 1 + (m_{12} - 29) \times 0.2682 + (m_{22} - 39) \times 0.3391 + \\
(m_{32} - 17) \times 0.4121 + (m_{42} - 17) \times 0.4795 + \delta_1 = 1 \\
1 + (m_{12} - 34) \times 0.2758 + (m_{22} - 20) \times 0.3632 + \\
(m_{32} - 12) \times 0.4561 + (m_{42} - 18) \times 0.5766 + \delta_2 = 1 \\
3m_{12} + 4m_{22} + 5m_{32} + 6m_{42} \leq 300 \\
2 \leq m_{12} \leq 73 \\
2 \leq m_{22} \leq 28 \\
2 \leq m_{32} \leq 23 \\
2 \leq m_{42} \leq 29 \\
\delta_1 \geq 0 \\
\delta_2 \geq 0 \\
\text{and } m_{h2} \text{ integers; } h=1,2,\dots,4
\end{array}$$

By solving the FGP model by software LINGO, we get the optimal solution as:

$$m_{12}^* = 44, m_{22}^* = 12, m_{32}^* = 6, m_{42}^* = 23 \text{ with } Z_1^* = 10.01193 \text{ and } Z_2^* = 8.852199$$

## 2.7 Conclusion

In this chapter, we have considered a multivariate stratified population with unknown strata weights. An optimum sampling design is proposed in the presence of non-response to estimate the unknown population means using double stratified sampling (DSS) strategy. The problem is formulated as a multi-objective integer nonlinear programming problem (MOINLPP). The problem turns out to be a non-linear bilevel programming problem. Then a fuzzy goal programming approach is used to solve the non-linear bi-level programming problem. The objective function at each level is non-linear in nature and there is one linear constraint with some upper and lower bounds. A compromise optimum allocation has been obtained in the minimum number of steps.



## **CHAPTER – III**

### **A Fuzzy Goal Programming Approach in Stochastic Multivariate Stratified Sample Surveys**

## CHAPTER - III

### A FUZZY GOAL PROGRAMMING APPROACH IN STOCHASTIC MULTIVARIATE STRATIFIED SAMPLE SURVEYS

#### 3.1 Introduction

Fuzzy programming is based on the basic idea to determine a feasible solution that minimizes the largest weighted deviation from any goal. This is an optimization programme. It can be thought of as an extension or generalization of linear programming to handle multiple, normally conflicting objective measures. The use of the fuzzy set theory for decision problems with several conflicting objectives was first introduced by Zimmermann (1978). Thereafter, various versions of fuzzy programming (FP) have been investigated and widely circulated in literature. The use of the concept of membership function of fuzzy set theory for satisfactory decisions was first introduced by Lai in 1996. To formulate the FGP Model of the problem, the fuzzy goals of the objectives are determined by determining individual optimal solution. The fuzzy goals are then characterized by the associated membership functions which are transformed into linear membership functions by first order Taylor series. Recently many authors have discussed fuzzy goal programming approach in different fields, some of them are Parra *et al.* (2001) who use this approach to portfolio selection problem, Sharma *et al.* (2007) work in the field of agriculture land allocation problems, Pramanik *et al.* (2011) apply FGP approach to quadratic bilevel multi-objective programming problem (QBLMPP), Paruang *et al.* (2012) presents FGP model for machine loading problem and minimize an average machine error and the total setup time, Pramanik & Banerjee (2012) in transportation, Haseen *et al.* (2012) and Gupta *et al.* (2012) in sample surveys etc.

The problem of allocation for a multivariate stratified survey becomes complicated because an allocation that is optimal for one characteristic is usually far from optimal for other characteristics unless the characteristics are highly correlated. In such situations, i.e. in multivariate stratified surveys, we need a compromise criterion that gives an allocation which is optimum for all characteristics in some sense and we have to consider the allocation problem as a multi-objective non linear programming problem (MONLPP) in which individual variances are to be minimized simultaneously subject to the cost constraint. Such an allocation may be called a "Compromise Allocation". Many authors have discussed the multivariate sample

allocation problem. Among them are Kozak (2006), Diaz-Garcia and Cortez (2008), Khan *et al.* (2010), Khowaja *et al.* (2011). Diaz-Garcia *et al.* (2007) dealt with the case when sampling variances are random in the constraints. Javaid and Bakhshi (2009) considered the case of random costs and used modified E- model for solving the problem. Bakhshi *et al.* (2010) find the optimal sample numbers in multivariate stratified sampling with a probabilistic cost constraint. Recently, some other authors who discuss stochastic programming in sample surveys are Ghufuran *et al.* (2011), Khan *et al.* (2011, 2012), Ali *et al.* (2013), Gupta *et al.* (2013), Raghav *et al.* (2014) etc.

In this chapter, the problem of obtaining the optimum compromise allocation has been formulated as a multi-objective non linear programming problem (MONLPP) and a fuzzy goal programming (FGP) approach is used to work out the compromise allocation in multivariate stratified sample surveys in which we define the membership functions of each objective function and then transformed the membership functions into equivalent linear membership functions with the help of first order Taylor series and finally forming the fuzzy goal programming model the desired compromise allocations are obtained. A numerical example is also worked out to illustrate the computational details of the proposed approach.

### 3.2 Formulations of the Problem

Consider a multivariate population consisting of  $N$  units which is divided into  $L$  disjoint strata of sizes  $N_1, N_2, \dots, N_L$  such that  $N = \sum_{h=1}^L N_h$ . Suppose that  $p$  characteristics ( $j = 1, \dots, p$ ) are measured on each unit of the population. We assume that the strata boundaries are fixed in advance. Let  $n_h$  units be drawn without replacement from the  $h^{th}$  stratum  $h = 1, \dots, L$ . For  $j^{th}$  character, an unbiased estimate of the population mean  $\bar{Y}_j$  ( $j = 1, \dots, p$ ), denoted by  $\bar{y}_{jst}$ , has its sampling variance

$$V(\bar{y}_{jst}) = \sum_{h=1}^L \left( \frac{1}{n_h} - \frac{1}{N_h} \right) W_h^2 S_{hj}^2, \quad j = 1, \dots, p, \quad (3.1)$$

where  $W_h = \frac{N_h}{N}$  is the stratum weight and  $S_{hj}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{hji} - \bar{Y}_j)^2$  is the variance for the  $j^{th}$  character in the  $h^{th}$  stratum. Let  $C$  be the upper limit on the total

cost of the survey. The problem of optimal sample allocation involves determining the sample sizes  $n_1, n_2, \dots, n_L$  that minimize the variances of various characters under the given sampling budget  $C$ . Within any stratum the linear cost function is appropriate when the major item of cost is that of taking the measurements on each unit. If travel costs between units in a given stratum are substantial, empirical and mathematical studies indicate that the costs are better represented by the expression  $\sum_{h=1}^L t_h \sqrt{n_h}$ ,

where  $t_h$  is the travel cost incurred in enumerating a sample unit in the  $h^{th}$  stratum (Beardwood *et al.* (1959)).

Assuming this non linear cost function one should have

$$C = c_0 + \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} \quad (3.2)$$

where  $c_h$ ;  $h=1,2,\dots,L$  denote the per unit cost of measurement in the  $h^{th}$  stratum,  $t_h$  is the travel cost for enumerating on a unit the  $j^{th}$  character in the  $h^{th}$  stratum and  $c_0$  is the overhead cost.

The restrictions  $2 \leq n_h \leq N_h$ ;  $h=1,2,\dots,L$  are introduced to obtain the estimates of the stratum variances and to avoid the problem of oversampling.

Thus the problem with non-linear cost function and ignoring the term independent of  $n_h$ , the allocation problem can be written as the following problem:

$$\left. \begin{array}{l} \min \quad \sum_{h=1}^L \frac{W_h^2 S_{jh}^2}{n_h} \\ \text{subject to} \quad \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0 \leq C \\ \text{and} \quad 2 \leq n_h \leq N_h; \quad h=1,2,\dots,L \end{array} \right\} j=1,2,\dots,p \quad (3.3)$$

In many practical situations the measurement cost  $c_h$  and the travel cost  $t_h$  in the various strata are not fixed and may be considered as random. Let us assume that  $c_h$  and  $t_h$ ,  $h=1,\dots,L$  are independently normally distributed random variables.

The formulated MONLPP (3.3) can be written in the following chance constrained programming form as:

$$\left. \begin{array}{l} \min \sum_{h=1}^L \frac{W_h^2 S_{jh}^2}{n_h} \\ \text{subject to } P\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0 \leq C\right) \geq p_0 \\ \text{and } 2 \leq n_h \leq N_h; \quad h = 1, 2, \dots, L \end{array} \right\}, j = 1, 2, \dots, p \quad (3.4)$$

where  $p_0$ ,  $0 < p_0 \leq 1$  is a specified probability.

The costs  $c_h$  and  $t_h$ ,  $h = 1, \dots, L$  have been assumed to be independently normally distributed random variables. Then the function defined in (3.2), will also be normally distributed with mean  $E\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0\right)$  and variance

$$V\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0\right).$$

If  $c_h \sim N(\mu_{ch}, \sigma_{ch}^2)$  and  $t_h \sim N(\mu_{th}, \sigma_{th}^2)$ , then the mean of the function

$\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0\right)$  is obtained as:

$$\begin{aligned} E\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0\right) &= E\left(\sum_{h=1}^L c_h n_h\right) + E\left(\sum_{h=1}^L t_h \sqrt{n_h}\right) + c_0 \\ &= \sum_{h=1}^L n_h E(c_h) + \sum_{h=1}^L \sqrt{n_h} E(t_h) + c_0 \\ &= \sum_{h=1}^L n_h \mu_{ch} + \sum_{h=1}^L \sqrt{n_h} \mu_{th} + c_0 \end{aligned} \quad (3.5)$$

and the variance as:

$$\begin{aligned} V\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0\right) &= V\left(\sum_{h=1}^L c_h n_h\right) + V\left(\sum_{h=1}^L t_h \sqrt{n_h}\right) + c_0 \\ &= \sum_{h=1}^L n_h^2 E(c_h^2) + \sum_{h=1}^L n_h E(t_h^2) + c_0 \\ &= \sum_{h=1}^L n_h^2 \sigma_{ch}^2 + \sum_{h=1}^L n_h \sigma_{th}^2 + c_0 \end{aligned} \quad (3.6)$$

Let  $f(t) = \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0$ , then the chance constraint in (3.4) is given by

$$\begin{aligned} P(f(t) \leq C) &\geq p_0, \\ \text{or } P\left\{\frac{f(t) - E(f(t))}{\sqrt{V(f(t))}} \leq \frac{C - E(f(t))}{\sqrt{V(f(t))}}\right\} &\geq p_0, \end{aligned}$$

where  $\frac{f(t) - E(f(t))}{\sqrt{V(f(t))}}$  is a standard normal variate with mean zero and variance one.

Thus the probability of realizing  $f(t)$  less than or equal to  $C$  can be written as:

$$P(f(t) \leq C) = \phi \left\{ \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right\} \quad (3.7)$$

where  $\phi(z)$  represents the cumulative density function of the standard normal variable evaluated at  $z$ . If  $K_\alpha$  represents the value of the standard normal variate at which  $\phi(K_\alpha) = p_0$ , then the constraint (3.7) can be written as

$$\phi \left\{ \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right\} \geq \phi(K_\alpha) \quad (3.8)$$

The inequality will be satisfied only if

$$\left\{ \frac{C - E(f(t))}{\sqrt{V(f(t))}} \right\} \geq (K_\alpha)$$

Or equivalently,

$$E(f(t)) + K_\alpha \sqrt{V(f(t))} \leq C \quad (3.9)$$

Substituting from (3.5) and (3.6) in (3.9), we get

$$\left( \sum_{h=1}^L n_h \mu_{ch} + \sum_{h=1}^L \sqrt{n_h} \mu_{th} + c_0 \right) + K_\alpha \sqrt{\sum_{h=1}^L n_h^2 \sigma_{ch}^2 + \sum_{h=1}^L n_h \sigma_{th}^2} \leq C \quad (3.10)$$

Since the constants  $\mu_{ch}, \mu_{th}, \sigma_{ch}$  and  $\sigma_{th}$  in (3.10) are unknown (by hypothesis). So

we will use the estimators of mean  $E \left( \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0 \right)$  and variance

$V \left( \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0 \right)$  given by

$$\hat{E} \left( \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} + c_0 \right) = \sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0$$

and

$$\hat{V}\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0\right) = \left(\sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h\right), \text{ say}$$

where  $\bar{c}_h, \bar{t}_h, \sigma_{ch}^2$  and  $\sigma_{th}^2$  are the estimated means and variances from the sample.

Thus an equivalent deterministic constraint to the stochastic constraint is given by

$$\left(\sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0\right) + K_\alpha \sqrt{\left(\sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h\right)} \leq C \quad (3.11)$$

Now in multivariate stratified sample surveys the problem of allocation with  $p$  independent characteristics is formulated as a multi-objective nonlinear programming problem (MONLPP). The objective is to minimize the individual variances of the estimates of the population means of  $p$  characteristics simultaneously, subject to the non linear probabilistic cost constraint. The formulated problem is given as:

$$\left. \begin{array}{l} \min \quad \left( \begin{array}{c} V(\bar{y}_{1st}) \\ \vdots \\ V(\bar{y}_{pst}) \end{array} \right) \\ \text{subject to } \hat{E}\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0\right) + K_\alpha \sqrt{\hat{V}\left(\sum_{h=1}^L c_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h}\right)} \leq C \\ \quad \quad \quad 2 \leq n_h \leq N_h \\ \text{and} \quad \quad n_h \text{ are integers; } h = 1, 2, \dots, L. \end{array} \right\} \quad (3.12)$$

To solve the problem (3.12) using stochastic programming, we first solve the following  $p$  non linear programming problems (NLPPs) for all the ' $p$ ' characteristics separately. The equivalent deterministic non linear programming problem to the stochastic programming problem is given by

$$\left. \begin{array}{l} \min \quad \sum_{h=1}^L \frac{W_h^2 S_{jh}^2}{n_h} \\ \text{subject to } \left(\sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0\right) + K_\alpha \sqrt{\left(\sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h\right)} \leq C \\ \quad \quad \quad 2 \leq n_h \leq N_h \\ \text{and} \quad \quad n_h \text{ are integers; } h = 1, 2, \dots, L. \end{array} \right\} \quad (3.13)$$

Let  $\underline{n}_{jh}^* = (n_{j1}^*, n_{j2}^*, \dots, n_{jL}^*)$  denote the solution to the  $j^{th}$  NLPP in (3.6) with  $V_j^*$  as the value of the objective function given by

$$V_j^* = \sum_{h=1}^L \frac{W_h^2 S_{hj}^2}{n_{hj}}; \quad j = 1, 2, \dots, p \quad (3.14)$$

A reasonable criterion to work out a compromise allocation may be to 'Minimize the sum of the variances  $V_j$ ;  $j = 1, 2, \dots, p$ '. But in this paper a new approach called "Fuzzy Goal Programming" is used to obtain a compromise allocation and discussed in next section.

### 3.3 Compromise solution using Fuzzy Goal Programming

Present approach is discussed by Pramanik *et al.* (2011) and Pramanik and Banerjee (2012) and here the approach is used in accordance with the above formulated problem. We now formulate the fuzzy programming model of multi-objective programming problem by transforming the objective functions  $V_1, V_2, \dots, V_j$ ;  $j = 1, 2, \dots, p$  into fuzzy goals by means of assigning an imprecise aspiration level to each of them. Let  $V_1^*, V_2^*, \dots, V_j^*$  be the optimal solutions of the each objective functions when calculated in isolation subject to the system constraints. Then the fuzzy goals appear in the form:  $V_j \gtrsim V_j^*$ ;  $j = 1, 2, \dots, p$

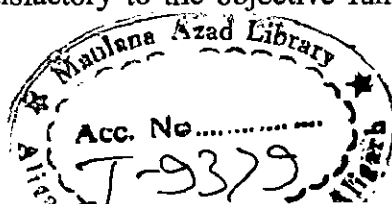
Using the individual best solutions, we formulate a payoff matrix as follows:

$$\begin{matrix} & V_1(\bar{n}) & \dots & V_2(\bar{n}) \\ \begin{matrix} (n_{1h}^*) \\ \vdots \\ (n_{jh}^*) \end{matrix} & \begin{bmatrix} V_1(n_{1h}^*) & \dots & V_j(n_{1h}^*) \\ \vdots & \ddots & \vdots \\ V_1(n_{jh}^*) & \dots & V_j(n_{jh}^*) \end{bmatrix} \end{matrix}, \text{ where } h = 1, 2, \dots, L \text{ and } j = 1, 2, \dots, p$$

where  $n_{jh}^*$ ,  $j = 1, 2, \dots, p$  i.e. the individual optimal points of each objective functions.

The maximum value of each column gives the upper tolerance limit for the objective functions and the minimum value of each column gives lower tolerance limit for the objective functions respectively.

The objective value, which is equal to or larger than  $V_j^*$  should be absolutely satisfactory to the objective functions. If the individual best solutions are identical,





then a satisfactory optimal solution of the system is reached. However, this situation arises rarely because the objectives are conflicting in general.

The non linear membership function  $\mu_j(\bar{n})$ ,  $j = 1, 2, \dots, p$  corresponding to the objective function  $V_j(\bar{n})$ ,  $j = 1, 2, \dots, p$  can be formulated as:

$$\mu_j(\bar{n}) = \begin{cases} 0, & \text{if } V_j(\bar{n}) \geq V_j^u(\bar{n}) \\ 1 - \frac{V_j(\bar{n}) - V_j^L(\bar{n})}{V_j^u(\bar{n}) - V_j^L(\bar{n})}, & \text{if } V_j^L(\bar{n}) \leq V_j(\bar{n}) \leq V_j^u(\bar{n}), j = 1, 2, \dots, p \\ 1, & \text{if } V_j(\bar{n}) \leq V_j^L(\bar{n}) \end{cases}$$

Here  $V_j^u(\bar{n})$  and  $V_j^L(\bar{n})$  are the upper and lower tolerance limits of the fuzzy objective goals.

Now the problem can be given as:

$$\left. \begin{aligned} & \max \mu_j(\bar{n}) \\ & \text{subject to } \left( \sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0 \right) + k_\alpha \sqrt{\left( \sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h \right)} \leq C \\ & \quad 2 \leq n_h \leq N_h \\ & \text{and } n_h \text{ integers; } h = 1, 2, \dots, L; j = 1, 2, \dots, p \end{aligned} \right\} \quad (3.15)$$

### 3.3.1 Linearization of the non linear membership functions by first order Taylor Series

Let  $\bar{n}_h^{(j)*}$ ,  $j = 1, 2, \dots, p$ ;  $h = 1, 2, \dots, L$  be the individual best solutions of the non linear membership functions subject to the constraints. Now, we transform the non-linear membership functions  $\mu_j(\bar{n})$ ,  $j = 1, 2, \dots, p$  into equivalent linear membership functions at individual best solution point by first order Taylor series as follows:

$$\mu_j(\bar{n}) \cong \mu_j(\bar{n}_h^{(j)*}) + (n_1 - n_1^{(j)*}) \frac{\partial}{\partial n_1} \mu_j(\bar{n}_h^{(j)*}) + \dots + (n_L - n_L^{(j)*}) \frac{\partial}{\partial n_L} \mu_j(\bar{n}_h^{(j)*}) = \xi_j(\bar{n})$$

### 3.3.2 Fuzzy Goal Programming model of Multi-objective NLPP

The NLPP represented by (3.15) reduces to the following problem

$$\left. \begin{aligned}
& \max \xi_j(\bar{n}) \\
& \text{subject to } \left( \sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0 \right) + k_\alpha \sqrt{\left( \sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h \right)} \leq C \\
& \quad 2 \leq n_h \leq N_h \\
& \text{and } n_h \text{ integers; } h = 1, 2, \dots, L; j = 1, 2, \dots, p
\end{aligned} \right\} \quad (3.16)$$

The maximum value of a membership function is unity (one), so for the defined membership functions in (3.16), the flexible membership goals having the aspiration level unity can be presented as:

$$\xi_j(\bar{n}) + \delta_j, j = 1, 2, \dots, p$$

Here  $\delta_j, j = 1, 2, \dots, p$  present the deviational variables.

Then our fuzzy goal programming (FGP) model is given as:

$$\left. \begin{aligned}
& \min \sum_{j=1}^p \delta_j \\
& \text{subject to } \xi_j + \delta_j = 1; j = 1, 2, \dots, p \\
& \quad \left( \sum_{h=1}^L \bar{c}_h n_h + \sum_{h=1}^L \bar{t}_h \sqrt{n_h} + c_0 \right) + K_\alpha \sqrt{\left( \sum_{h=1}^L \sigma_{ch}^2 n_h^2 + \sum_{h=1}^L \sigma_{th}^2 n_h \right)} \leq C \\
& \quad 2 \leq n_h \leq N_h \\
& \quad \delta_j \geq 0 \\
& \text{and } n_h \text{ are integers; } h = 1, 2, \dots, L.
\end{aligned} \right\} \quad (3.17)$$

### 3.4 Numerical Illustration

In the table below the stratum sizes, stratum weights, stratum standard deviations, measurement costs, and the travel costs within stratum are given for four different characteristics under study in a population stratified in five strata. The data are mainly from Chatterjee (1968). The values of strata sizes are added assuming the population size as 6000. The total budget of the survey is assumed to be 1500 units with an overhead cost = 300 units.

**Table 1: Values of  $N_h, W_h, c_h, t_h$  and  $S_{jh}$  for five strata and four characteristics:**

$h$	$N_h$	$W_h$	$S_{jh}$			
			$S_{1h}$	$S_{2h}$	$S_{3h}$	$S_{4h}$
1	1500	0.25	28	206	38	120
2	1920	0.32	24	133	26	184
3	1260	0.21	32	48	44	173
4	480	0.08	54	37	78	92
5	840	0.14	67	9	76	117

In this problem  $c_1, c_2, c_3, c_4, c_5, t_1, t_2, t_3, t_4$  and  $t_5$  are independently normally distributed random variables with known means and standard deviations

$$E(c_1)=1, E(c_2)=1, E(c_3)=1.5, E(c_4)=1.5 \text{ and } E(c_5)=2$$

$$E(t_1)=0.5, E(t_2)=0.5, E(t_3)=1, E(t_4)=1 \text{ and } E(t_5)=1.5$$

$$V(c_1)=0.25, V(c_2)=0.25, V(c_3)=0.35, V(c_4)=0.35 \text{ and } V(c_5)=0.45$$

$$V(t_1)=0.125, V(t_2)=0.125, V(t_3)=0.175, V(t_4)=0.175 \text{ and } V(t_5)=0.225$$

Using the values given in Table 1 the MONLPP (3.12) and their optimal solutions  $\underline{n}_h^*$ ;  $j=1,2,\dots,5$  with the corresponding values of  $V_j^*$  are listed below in Eqn. (3.18).

These values are obtained by software LINGO.

$$\left. \begin{aligned}
\min V_1 &= \frac{49}{n_1} + \frac{58.9824}{n_2} + \frac{45.1584}{n_3} + \frac{18.6624}{n_4} + \frac{87.9844}{n_5} \\
\min V_2 &= \frac{2652.25}{n_1} + \frac{1811.3536}{n_2} + \frac{101.6064}{n_3} + \frac{8.7616}{n_4} + \frac{1.5876}{n_5} \\
\min V_3 &= \frac{90.25}{n_1} + \frac{69.2224}{n_2} + \frac{85.3776}{n_3} + \frac{38.9376}{n_4} + \frac{113.2096}{n_5} \\
\min V_4 &= \frac{900}{n_1} + \frac{3466.8544}{n_2} + \frac{1319.8689}{n_3} + \frac{54.1696}{n_4} + \frac{268.9844}{n_5} \\
\text{subject to } &\begin{pmatrix} 1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5 \\ + 0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5} \end{pmatrix} \\
&+ 2.33 \sqrt{\left\{ \begin{aligned} &(0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) + \\ &(0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5) \end{aligned} \right\}} \leq 1200 \\
&\text{and} \\
&2 \leq n_1 \leq 1500 \\
&2 \leq n_2 \leq 1920 \\
&2 \leq n_3 \leq 1260 \\
&2 \leq n_4 \leq 480 \\
&2 \leq n_5 \leq 840
\end{aligned} \right\} \quad (3.18)$$

The optimum allocation  $\underline{n}_1^* = (n_{11}^*, n_{12}^*, n_{13}^*, n_{14}^*, n_{15}^*)$  is  $n_{11}^* = 132.999, n_{12}^* = 143.2324, n_{13}^* = 107.7228, n_{14}^* = 72.3840$  and  $n_{15}^* = 127.6964$

The corresponding value of the variance ignoring finite population correction (fpc) is  $V_1^* = 2.148212$ .

The optimum allocation  $\underline{n}_2^* = (n_{21}^*, n_{22}^*, n_{23}^*, n_{24}^*, n_{25}^*)$  is  $n_{21}^* = 303.1810, n_{22}^* = 259.2840, n_{23}^* = 60.5848, n_{24}^* = 18.3975$  and  $n_{25}^* = 6.6782$

The corresponding value of the variance ignoring finite population correction (fpc) is  $V_2^* = 18.12507$ .

The optimum allocation  $\underline{n}_3^* = (n_{31}^*, n_{32}^*, n_{33}^*, n_{34}^*, n_{35}^*)$  is

$n_{31}^* = 142.0023, n_{32}^* = 126.7286, n_{33}^* = 117.2123, n_{34}^* = 82.6231$  and  $n_{35}^* = 1176.3308$

The corresponding value of the variance ignoring finite population correction (fpc) is  $V_3^* = 3.346324$ .

The optimum allocation  $\underline{n}_4^* = (n_{41}^*, n_{42}^*, n_{43}^*, n_{44}^*, n_{45}^*)$  is

$$n_{41}^* = 139.7336, n_{42}^* = 246.2649, n_{43}^* = 139.3793, n_{44}^* = 31.8239 \text{ and } n_{45}^* = 59.5315$$

The corresponding value of the variance ignoring finite population correction (fpc) is  $V_4^* = 36.19729$ .

Now the payoff matrix is

$$\text{Payoff matrix} = \begin{bmatrix} 2.146266 & 33.66483 & 33.66483 & 3.378915 \\ 15.32374 & 18.12512 & 21.04245 & 81.24543 \\ 2.171515 & 33.95712 & 3.346323 & 47.89731 \\ 2.978546 & 27.36704 & 4.664726 & 36.19729 \end{bmatrix}$$

Here the upper and lower tolerance limits can be given as:

$$V_1^u = 15.32374, V_1^L = 2.146266$$

$$V_2^u = 33.95712, V_2^L = 18.12512$$

$$V_3^u = 21.04245, V_3^L = 3.346323$$

$$V_4^u = 81.24543, V_4^L = 36.19729$$

The non-linear membership functions can be formulated as:

$$\mu_1(\bar{n}) = 1 - \frac{V_1(\bar{n}) - 2.146266}{15.32374 - 2.146266}$$

$$\mu_2(\bar{n}) = 1 - \frac{V_2(\bar{n}) - 18.12512}{33.95712 - 18.12512}$$

$$\mu_3(\bar{n}) = 1 - \frac{V_3(\bar{n}) - 3.346323}{21.04245 - 3.346323}$$

$$\mu_4(\bar{n}) = 1 - \frac{V_4(\bar{n}) - 36.19729}{81.24543 - 36.19729}$$

The membership function  $\mu_1(\bar{n})$  is maximal at the point (132.999, 143.2324, 107.7228, 72.3840, 127.6964),  $\mu_2(\bar{n})$  is maximal at the point (303.1810, 259.2840, 60.5848, 18.3975, 6.6782),  $\mu_3(\bar{n})$  is maximal at the point (142.0023, 126.7286, 117.2123, 82.6231, 117.3308) and  $\mu_4(\bar{n})$  is maximal at the point (139.7336, 246.2649, 139.3793, 31.8239, 59.5315) respectively.

Then, the non-linear membership functions are transformed into linear at the individual best solution point by first order Taylor polynomial series as follows:

$$\begin{aligned}
\mu_1(\bar{n}) &\cong 1 + (n_1 - 132.999) \times 0.0002 + (n_2 - 143.2324) \times 0.0002 + (n_3 - 107.7228) \times 0.0003 \\
&\quad + (n_4 - 72.3840) \times 0.0003 + (n_5 - 127.6964) \times 0.0003 = \xi_1(\bar{n}) \\
\mu_2(\bar{n}) &\cong 1 + (n_1 - 303.1810) \times 0.0018 + (n_2 - 259.2840) \times 0.0017 + (n_3 - 60.5848) \times 0.0017 \\
&\quad + (n_4 - 18.3975) \times 0.0016 + (n_5 - 6.6782) \times 0.0023 = \xi_2(\bar{n}) \\
\mu_3(\bar{n}) &\cong 1 + (n_1 - 142.0023) \times 0.0003 + (n_2 - 126.7286) \times 0.0002 + (n_3 - 117.2123) \times 0.0004 \\
&\quad + (n_4 - 82.6231) \times 0.0003 + (n_5 - 117.3308) \times 0.0003 = \xi_3(\bar{n}) \\
\mu_4(\bar{n}) &\cong 1 + (n_1 - 139.7336) \times 0.0010 + (n_2 - 246.2649) \times 0.0013 + (n_3 - 139.3793) \times 0.0015 \\
&\quad + (n_4 - 31.8239) \times 0.0012 + (n_5 - 59.5315) \times 0.0017 = \xi_4(\bar{n})
\end{aligned}$$

Then, the FGP model for solving MONLPP (3.17) is given in Eqn. (3.19) as follows:

$$\left. \begin{aligned}
&\min \quad \sum_{j=1}^4 \delta_j \\
&\text{subject to} \\
&1 + (n_1 - 132.999) \times 0.0002 + (n_2 - 143.2324) \times 0.0002 + (n_3 - 107.7228) \times 0.0003 + \\
&\quad (n_4 - 72.3840) \times 0.0003 + (n_5 - 127.6964) \times 0.0004 + \delta_1 = 1 \\
&1 + (n_1 - 303.1810) \times 0.0018 + (n_2 - 259.2840) \times 0.0017 + (n_3 - 60.5848) \times 0.0017 + \\
&\quad (n_4 - 18.3975) \times 0.0016 + (n_5 - 6.6782) \times 0.0023 + \delta_2 = 1 \\
&1 + (n_1 - 142.0023) \times 0.0003 + (n_2 - 126.7286) \times 0.0002 + (n_3 - 117.2123) \times 0.0004 + \\
&\quad (n_4 - 82.6231) \times 0.0003 + (n_5 - 117.3308) \times 0.0005 + \delta_3 = 1 \\
&1 + (n_1 - 139.7336) \times 0.0010 + (n_2 - 246.2649) \times 0.0013 + (n_3 - 139.3793) \times 0.0015 + \\
&\quad (n_4 - 31.8239) \times 0.0012 + (n_5 - 59.5315) \times 0.0017 + \delta_4 = 1 \\
&(1n_1 + 1n_2 + 1.5n_3 + 1.5n_4 + 2n_5 + 0.5\sqrt{n_1} + 0.5\sqrt{n_2} + 1\sqrt{n_3} + 1\sqrt{n_4} + 1.5\sqrt{n_5}) + 2.33 \\
&\quad \sqrt{(0.25n_1^2 + 0.25n_2^2 + 0.35n_3^2 + 0.35n_4^2 + 0.45n_5^2) +} \\
&\quad \sqrt{(0.125n_1 + 0.125n_2 + 0.175n_3 + 0.175n_4 + 0.225n_5)} \leq 1200 \\
&2 \leq n_1 \leq 1500; 2 \leq n_2 \leq 1920; 2 \leq n_3 \leq 1260; 2 \leq n_4 \leq 480; 2 \leq n_5 \leq 840 \\
&\delta_j \geq 0 \quad \text{and} \quad n_h \text{ are integers; } h = 1, 2, \dots, L; \quad j = 1, \dots, 4
\end{aligned} \right\}$$

By solving the FGP model by software LINGO, we get the optimal solution as:

$n_1 = 198, n_2 = 214, n_3 = 95, n_4 = 37$  and  $n_5 = 79$  with a total of 623. Corresponding to this allocation the values of the variances for the four characters are obtained as  $V_1 = 2.616561, V_2 = 23.18591, V_3 = 4.163389, V_4 = 39.49937$  with the total cost consumption for conducting the survey i.e.  $C = 1200$  units.

## 5. Conclusion

This chapter provides the description of the multivariate stratified sampling problem with non linear objective function and a probabilistic non linear cost constraint. The problem is formulated as a multi-objective non-linear programming problem (MONLPP). The fuzzy goal programming (FGP) approach has been used to solve the stochastic multivariate stratified sampling with non-linear objective function and probabilistic non-linear cost constraint which is formulated as an MONLPP. In the model formulation of the problem, we first determine the individual best solution of the objective functions subject to the system constraints and construct the non-linear membership functions of each objective. The non linear membership functions are then transformed into equivalent linear membership functions by first order Taylor series at the individual best solution point. Fuzzy goal programming approach is then used to achieve maximum degree of each of the membership goals by minimizing negative deviational variables and finally obtain the compromise allocation.



## **CHAPTER – IV**

### **Geometric Programming Approach in Three – Stage Sampling Design**



## CHAPTER – IV

### GEOMETRIC PROGRAMMING APPROACH IN THREE - STAGE SAMPLING DESIGN

#### 4.1 Introduction

The specified level of precision is attained by an optimum allocation with the minimum cost. The use of three stage sampling designs generally specifies three stages of selection: primary sampling units (PSUs) at the first stage, sub samples from PSUs at second stage as a secondary sampling units (SSUs) units and again sub samples from SSUs at third stage as a tertiary sampling units (TSUs). The three stage sampling designs are well analyzed when two variable is measured. Different methods are available for obtaining the optimum allocation of sampling units to each stage. The problem of optimum allocation in three-stage sampling with two characters is described in standard texts on sampling (see Cochran (1977)). For instance, in surveys to estimate crop production in India (Sukhatme, 1947), the village is a convenient sampling unit. Within a village, only some of the fields growing the crop in question are selected, so that the field is a sub-unit. When a field is selected, only certain parts of it are cut for the determination of yield per acre; thus the sub unit itself is sampled. Here we have to find the optimal sample sizes  $n$ ,  $m$  and  $p$  for all the three stages with the minimum cost.

The estimation of optimal sample size for each characteristic is done one by one and then final sampling design can selected among the individual solutions. Practically it is impossible for optimum allocations from this procedure for individuals because an allocation, which is optimum for one characteristic, may not be optimum for other characteristic. It is very necessary to work out an acceptable sampling design for certain situations which will be optimal for all the characteristics.

Geometric programming (GP) is very much connected with geometrical concepts because this method based on geometric inequality. The sums and products of positive numbers are important properties of GP. The degree of difficulties in GP plays very important roles in the solution of mathematical programming problems. The degree of difficulty of a GP problem is defined as:

Degree of difficulty = total number of terms in the objective functions and  
constraints - total number of decision variables of the objective  
functions and constraints - 1

If the degree of difficulty of primal problem is zero, then unique dual feasible solution exists.

If the problem has positive degree of difficulty, then the objective function can be maximized by finding the dual feasible region, and if there is negative degree of difficulty then inconsistency of the dual constraints may occur.

Geometric Programming (GP), a smooth, systematic non-linear programs used for solving sampling problems, the engineering design problems that takes the form of mathematical programming problems. Duffin and Zener has done the work in the field of engineering design problems in the early 1960s, and further extended by Duffin *et al.* (1967). Engineering design problems was also solved by Shiang (2008) and Shaojian *et.al* (2008) with the help of GP. Davis and Rudolph (1987) applied GP to optimal allocation of integrated samples in quality control.

In this chapter, we have considered the problem of three-stage sampling design. The three-stage sampling design problem has been formulated as a non linear convex programming problem. We have developed a geometric programming technique for the solution of the resulting mathematical programming problem. Presently, the solution of the mathematical programming problems, sampling problems, engineering problems; Management problems *etc.* are very much dependent upon software. With this regard, we have provided an effective manual method as well as using LINGO software for obtaining optimum allocations in three-stage sampling with the help of primal-dual relationship of geometric programming. The manual description of the solution procedure of geometric programming is very simple in comparison to the complex analytical techniques used in statistical literature. The presentation of the paper is as follows: The formulation of an allocation problem in a three-stage sampling design is discussed in section 4.2 and the solution procedure for solving above formulated problem with geometric programming approach is discussed in section 4.3. The illustrative numerical example is then presented in section 4.4 and finally some comments and conclusions which are drawn from the discussion are given in section 4.5.

## 4.2 Formulation of the Problem:

Let us consider the population consists of  $NMK$  elements grouped into  $N$  first-stage units of  $M$  second-stage units and  $K$  third stage units each. Let  $n, m$  and  $k$  is the corresponding sample sizes selected with equal probability and without replacement at each stage. Let  $y_{iju}$  be the value obtained for  $u^{th}$  third-stage unit in the  $j^{th}$  second-stage unit drawn from the  $i^{th}$  primary unit. The relevant population means per third-stage unit are as follows:

$$\bar{y}_{ij} = \frac{\sum_u y_{iju}}{K} = \text{Sample mean per } j^{th} \text{ second-stage unit at the } i^{th} \text{ PSU.}$$

$$\bar{\bar{y}}_i = \frac{\sum_j \sum_u y_{iju}}{MK} = \text{Mean per element at the PSU.}$$

$$\bar{\bar{\bar{y}}} = \frac{\sum_i \sum_j \sum_u y_{iju}}{NMK} = \text{Mean per element in the population.}$$

The following population variances are required.

$$S_1^2 = \frac{\sum_i (Y_i - \bar{\bar{\bar{y}}})^2}{N-1} = \text{variance within PSU means.}$$

$$S_2^2 = \frac{\sum_i \sum_j (\bar{y}_{ij} - \bar{\bar{y}}_i)^2}{N(M-1)} = \text{variance among SSU within PSU means.}$$

$$S_3^2 = \frac{\sum_i \sum_j \sum_u (y_{ijk} - \bar{y}_{ij})^2}{NM(K-1)} = \text{variance for TSU among SSU within PSU means.}$$

In case of equal TSU an unbiased estimate of  $\bar{\bar{\bar{y}}}$  is  $y_i$  with its sampling variance as,

$$v(\bar{\bar{\bar{y}}}) = \frac{1-f_1}{n} S_1^2 + \frac{1-f_2}{nm} S_2^2 + \frac{1-f_3}{nmk} S_3^2 \quad (4.1)$$

In three- stage sampling the total cost function may be given by:

$$C = C_1 n + C_2 n m + C_3 n m k \quad (4.2)$$

$C_1$  = The cost for  $i^{th}$  PSU in the survey.

$C_2$  = The cost of enumerating the  $j^{th}$  character per element in the SSU

$C_3$  = The cost for finding the  $u^{th}$  character per element in the TSU

$C$  = the total cost of enumerating all the  $p$  characters per TSU.

Suppose that it is required to find the values of  $n, m$  and  $k$  so that the cost  $C$  is minimized, subject to the upper limits on the variances. If  $N, M$  and  $K$  are large, then from (4.2), the limits on the variances may be expressed as:

$$\frac{S_1^2}{n} + \frac{S_2^2}{nm} + \frac{S_3^2}{nmk} \leq v_j \quad j = 1, \dots, p \quad (4.3)$$

where,  $v_j$  are the upper limits on the variances of various characters. Here  $S_{bj}^2$  is the variance among primary stage units means,  $S_{wj}^2$  is the variance secondary stage units means, and  $S_{kj}^2$  is the variance of the tertiary stage units means for  $j^{th}$  characteristic respectively. The problem therefore reduces to find  $n, m$  and  $k$  which:

$$\left. \begin{array}{l} \min \quad C = C_1 n + C_2 n m + C_3 n m k \\ \text{subject to} \quad \left. \begin{array}{l} \frac{S_{bj}^2}{n} + \frac{S_{wj}^2}{nm} + \frac{S_{kj}^2}{nmk} \\ n \geq 1, \quad m \geq 1 \end{array} \right\} \end{array} \right\} \quad (4.4)$$

### 4.3 Geometric programming approach in three-stage sample surveys

Posynomial functions are minimized in the Geometric programming (GP) technique subject to several constraints. Posynomial functions can be defined as polynomials in several variables with positive coefficients in all terms and the power to which the variables are raised can be any real numbers. The cost function and the variance constraint functions are in the form of posynomials. Geometric programming always transforms the primal problem of minimizing a “posynomial” subject to “posynomial” constraints to a dual problem of maximizing a function of the weights on each constraint. Generally constraints are less than strata, so the transformation simplifies

the procedure. The mathematical form of problem (4.4) can be expressed in the following way:

$$\begin{aligned} \underline{X} &= (x_1, x_2, x_3) \text{ where } (x_1 = n, x_2 = nm \text{ and } x_3 = nmk) \\ \min C(X) &= \sum_{i=1}^3 C_i x_i = C_1 n + C_2 nm + C_3 nmk \quad (i) \\ \text{subject to } g(X) &= \sum_{i=1}^3 \frac{a_{iq}}{x_q} \leq v_q, \quad q = 1, \dots, p. \quad (ii) \\ \text{and } x_i &\geq 0, \quad i = 1, 2, 3. \quad (iii) \end{aligned} \quad (4.5)$$

After substituting in the above equations we get:

$$\begin{aligned} x_1 &= n, x_2 = nm, x_3 = nmk, \\ S_{bq}^2 &= a_{1q}, S_{wq}^2 = a_{2q}, S_{kq}^2 = a_{3q} \text{ for } q = 1, \dots, p \end{aligned}$$

In the above equations, we have noticed that the objective function 4.5 (i) is linear and the constraints 4.5 (ii) are nonlinear and the reduced standard GP (Primal) problem can be stated as:

$$\begin{aligned} \min & f_0(x) \\ \text{subject to } & f_q(x) \leq 1, \quad q = 1, \dots, p \\ \text{and } & x_j > 0, \quad j = 1, \dots, n \end{aligned} \quad (4.6)$$

The posynomial is defined as:

$$f_q(x) = \sum_{i \in [q]} d_i \left\{ \prod_{j=1}^n x_j^{p_{ij}} \right\}, d_i \geq 0, x_j > 0, q = 0, 1, \dots, p \quad (4.7)$$

where  $d_i = C_i$  and  $d_i = a_{ia}, i = 1, 2, 3$ .

The number of posynomial terms in the function can be denoted by  $k$ , the number of variables is denoted by  $n$  and the exponents  $p_{ij}$  are real constants. The objective function  $C(x)$  for our allocation problem that is given in 4.7 (i) and 4.7 (ii) has  $k = 3$ ,  $n = 3$

After substituting the values in the equations given below we get:

$$x_1 = n, x_2 = nm, x_3 = nmk, S_{bq}^2 = a_{1q}, S_{wq}^2 = a_{2q}, S_{kq}^2 = a_{3q} \text{ for } q = 1, 2, \dots, p$$

$p_{11} = p_{22} = p_{33} = -1$ ,  $p_{12} = p_{21} = 0$  and  $d_i = c_i$ ,  $i = 1, 2, 3$  and

the  $q^{th}$  constraint has  $k = 3$ ,  $n = 3$ ,  $p_{11} = p_{22} = -1$ , and  $d_i = a_{iq}$ ,  $i = 1, 2, 3$ .

The dual form of Geometric Programming problem which is stated in (4.6) can be rewritten as:

$$\left. \begin{array}{ll} \max \left[ \prod_{q=0}^p \prod_{i \in j[q]} \left( \frac{d_i}{w_i} \right)^{w_i} \right] \prod_{q=1}^p \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} & (i) \\ \text{subject to } \sum_{i \in [0]} w_i = 1 & (ii) \\ \sum_{q=0}^p \sum_{i \in j[q]} p_{ij} w_i = 0 & (iii) \\ w_i \geq 0, \quad q = 0, \dots, p \text{ and } i = 1, \dots, k_p. & (iv) \end{array} \right\} \quad (4.8)$$

Step 1: For the Optimum value of the objective function, the objective function always takes the form:

$$\begin{aligned} C_0(x^*) &= \left( \frac{\text{Coeffi. of first term}}{w_{01}} \right)^{w_{01}} \times \left( \frac{\text{Coeffi. of second term}}{w_{02}} \right)^{w_{02}} \\ &\times \dots \times \left( \frac{\text{Coeffi. of last term}}{w_{0k}} \right)^{w_{0k}} \left( \sum w' \text{ s in the first constraints} \right)^{\sum w' \text{ s in the first constraints}} \\ &\left( \sum w' \text{ s in the last constraints} \right)^{\sum w' \text{ s in the last constraints}} \end{aligned}$$

The objective function for our problem is:

$$Cost = \left( \frac{C_1}{w_1} \right)^{w_1} \left( \frac{C_2}{w_2} \right)^{w_2} \left( \frac{C_3}{w_3} \right)^{w_3} (k_1)^{w_4} (k_2)^{w_5} (k_3)^{w_6} \quad (4.9)$$

Where  $k_1 = \frac{a_1}{v_1}$ ,  $k_2 = \frac{a_2}{v_2}$ ,  $k_3 = \frac{a_3}{v_3}$

Step 2: The equations that can be used for geometric program for the weights are given below:

$$\sum w' \text{ s in the objective function} = 1 \quad (4.10)$$

and for each primal variable  $x_j$  given  $n$  variables and  $k$  terms

$$\sum_{i=1}^m (w_i \text{ for each trms } ) \times (\text{exponent on } x_j \text{ in that term}) = 0 \quad (4.11)$$

In our case:

$$w_1 + w_2 + w_3 = 1 \quad (\text{Normalization condition, from eqn.4.8 (ii)}) \quad (4.12)$$

$$\left. \begin{aligned} (1)w_1 + (0)w_2 + (0)w_3 + (-1)w_4 + (0)w_5 + (0)w_6 &= 0 \\ (0)w_1 + (1)w_2 + (0)w_3 + (0)w_4 + (-1)w_5 + (0)w_6 &= 0 \\ (0)w_1 + (0)w_2 + (1)w_3 + (0)w_4 + (0)w_5 + (-1)w_6 &= 0 \end{aligned} \right\} \quad (4.13)$$

Orthogonality conditions are represented in equation (4.13). Combindly, these conditions are referred to as dual constraints. For more details see Duffin *et al.* (1967). Now after solving equation (4.13) we get:

$$\begin{aligned} w_1 - w_4 &= 0 \Rightarrow w_1 = w_4 \\ w_2 - w_5 &= 0 \Rightarrow w_2 = w_5 \\ w_3 - w_6 &= 0 \Rightarrow w_3 = w_6 \end{aligned}$$

Step 3: The terms which are used in the constraints to the optimal solution are always proportional to their weights. This can be expressed as:

$$\left. \begin{aligned} \frac{k_1}{x_1} &= \frac{w_4}{w_4 + w_5 + w_6} = w_1 \\ \frac{k_2}{x_2} &= \frac{w_5}{w_4 + w_5 + w_6} = w_2 \\ \frac{k_3}{x_3} &= \frac{w_6}{w_4 + w_5 + w_6} = w_3 \end{aligned} \right\} \quad (4.14)$$

Step 4: The primal variables can be obtained as:

$$\begin{aligned} C_0(x^*) &= \frac{\text{first term in the objective function}}{w_1} \\ &= \frac{\text{second term in the objective objectiue function}}{w_2} \\ &= \dots = \frac{\text{last term in the objective function}}{w_k} \end{aligned}$$

In this case:

$$\frac{C_1 x_1}{w_1} = \frac{C_2 x_2}{w_2} = \frac{C_3 x_3}{w_3}, \text{ here } \left[ w_1 = \frac{k_1}{x_1} \right] \left[ w_2 = 1 - \frac{k_1}{x_1} - \frac{k_3}{x_3} \right], \left[ w_3 = 1 - \frac{k_1}{x_1} - \frac{k_2}{x_2} \right] \quad (4.15)$$

Now the above equations can be solved for obtaining the variables

$$\frac{k_1}{x_1} + \frac{k_2}{x_2} + \frac{k_3}{x_3} = 1$$

$$\frac{k_2}{x_2} = 1 - \frac{k_1}{x_1} - \frac{k_3}{x_3}$$

$$x_2 = \frac{k_2 x_2 x_3}{x_1 x_3 - k_1 x_3 - k_3 x_1} \quad (4.16)$$

$$\frac{c_1 x_1}{w_1} = \frac{c_2 x_2}{w_2}$$

$$\frac{c_1 x_1}{\frac{k_1}{x_1}} = \frac{c_2 x_2}{\frac{k_2}{x_2}} \Rightarrow x_1 = x_2 \sqrt{\left( \frac{c_2 k_1}{c_1 k_2} \right)} \quad (4.17)$$

$$\text{Let } A = \sqrt{\frac{k_1 c_2}{c_1 k_2}} \quad (*)$$

$$\frac{c_1 x_1}{w_1} = \frac{c_3 x_3}{w_3} \Rightarrow \frac{c_1 x_1}{\frac{k_1}{x_1}} = \frac{c_3 x_3}{\frac{k_3}{x_3}} \Rightarrow x_1 = x_3 \sqrt{\frac{c_3 k_1}{c_1 k_3}} \text{ let } B = \sqrt{\frac{c_3 k_1}{c_1 k_3}} \quad (**)$$

then, we have  $x_1 = B x_3$  (4.18), and by putting the values  $x_2$  from equation (4.16) in equation (4.17), we have

$$x_1 = (k_2 A + k_1) \frac{x_3}{(x_3 - k_3)} \quad (4.19)$$

Now the value of  $x_2$  is obtained from the above equation (4.16).

$$x_2 = \frac{(k_2 x_1 x_3)}{(x_1 x_3 - k_1 x_3 - k_3 x_1)} \quad (4.20)$$



Again after solving equation (4.19) for  $x_3$ , we get

$$x_3 = \left( \frac{Ax_2 k_3}{Ax_2 - Ak_2 - k_1} \right) \quad (4.21)$$

#### 4.4 Numerical illustration

For illustration we have considered the following hypothetical data:

$$S_{b_1}^2 = 0.4560, \quad S_{w_1}^2 = 0.8878, \quad S_{k_1}^2 = 0.9040$$

$$S_{b_2}^2 = 0.5234, \quad S_{w_2}^2 = 0.4410, \quad S_{k_2}^2 = 0.5503$$

$$S_{b_3}^2 = 0.4085, \quad S_{w_3}^2 = 0.1128, \quad S_{k_3}^2 = 0.2013$$

$$v_1 = 0.03110, \quad v_2 = 0.02820, \quad v_3 = 0.02013$$

$$C_1 = 10, \quad C_2 = 3, \quad C_3 = 1.5$$

$$\left. \begin{array}{l} \min = C_1 n + C_2 nm + C_3 nmk \\ \text{subject to } \left. \begin{array}{l} \frac{S_{b_1}^2}{x_1} + \frac{S_{w_1}^2}{x_2} + \frac{S_{k_1}^2}{x_3} \leq v_1 \\ \frac{S_{b_2}^2}{x_1} + \frac{S_{w_2}^2}{x_2} + \frac{S_{k_2}^2}{x_3} \leq v_2 \\ \frac{S_{b_3}^2}{x_1} + \frac{S_{w_3}^2}{x_2} + \frac{S_{k_3}^2}{x_3} \leq v_3 \\ x_1 \geq 1, \quad x_2 \geq 1, \quad x_3 \geq 1 \end{array} \right\} \end{array} \right\} \quad (4.22)$$

Now by using the above values in equation (4.22) we get:

$$\left. \begin{array}{l} \min C = 10x_1 + 3x_2 + 1.5x_3 \quad (i) \\ \text{subject to } \frac{0.4560}{x_1} + \frac{0.8878}{x_2} + \frac{0.9040}{x_3} \leq 0.03110 \quad (ii) \\ \frac{0.5234}{x_1} + \frac{0.4410}{x_2} + \frac{0.5503}{x_3} \leq 0.02820 \quad (iii) \\ \frac{0.4085}{x_1} + \frac{0.1128}{x_2} + \frac{0.2013}{x_3} \leq 0.02110 \quad (iv) \\ x_1, x_2, x_3 \geq 0 \quad (v) \end{array} \right\} \quad (4.23)$$

The normalized constraints for our problem are:

$$\left. \begin{aligned} \frac{0.4560}{0.03110} + \frac{0.8878}{0.03110} + \frac{0.9040}{0.03110} &\leq 1 \quad (i) \\ \frac{0.5234}{0.02820} + \frac{0.4410}{0.02820} + \frac{0.5503}{0.02820} &\leq 1 \quad (ii) \\ \frac{0.4085}{0.02110} + \frac{0.1128}{0.02110} + \frac{0.2013}{0.02110} &\leq 1 \quad (iii) \end{aligned} \right\} \quad (4.24)$$

The above equation will give the following:

$$\left. \begin{aligned} \frac{14.6624}{X_1} + \frac{28.5466}{X_2} + \frac{29.0675}{X_3} &\leq 1 \quad (i) \\ \frac{16.8296}{X_1} + \frac{14.18}{X_2} + \frac{17.6945}{X_3} &\leq 1 \quad (ii) \\ \frac{19.3601}{X_1} + \frac{5.3459}{X_2} + \frac{9.5403}{X_3} &\leq 1 \quad (iii) \end{aligned} \right\} \quad (4.25)$$

The constraint 4.24 (ii) is assumed to be active (if all the three constraints were active, then two of them will not be able for finding an optimal dual solution nor an optimal solution to the original problem).

**(Conditions for active and inactive constraints:** At any feasible point  $\underline{x}$  the  $h^{\text{th}}$  constraint is said to be active if  $\delta_h(x) = 0$  and inactive if  $\delta_h(x) > 0$ . In our case the constraint 4.24 (ii) is active because it satisfies the condition of active constraint. This can be explained as: After putting the value of  $x_1^*$ ,  $x_2^*$  and  $x_3^*$  in the equation 4.24 (ii) we get:

$$\frac{16.8296}{31.9740} + \frac{14.18}{53.5847} + \frac{17.6945}{84.6543} \leq 1 \Rightarrow .9998 - 1 = -.0002 \approx 0.)$$

Then  $K_1 = 16.8296$ ,  $K_2 = 14.18$  and  $K_3 = 17.6945$

On substituting the values of  $K_1, K_2, K_3, C_1, C_2$  and  $C_3$  in equations (4.16), (4.17), (\*), (\*\*), (4.18), (4.19), (4.20) and (4.21), we get the values of  $x_1, x_2$  and  $x_3$  as:

$$x_1 = 31.9740, \quad x_2 = 53.5847 \text{ and } x_3 = 84.6543.$$

By rounding the above values we get:

$$x_1^* = 32, x_2^* = 54 \text{ and } x_3^* = 85$$

The optimum values of the sample sizes can be obtained as:

$$x_1 = n = 31.9740 \cong 32, x_2 = nm = 53.5847 \Rightarrow m = \frac{53.5847}{31.9740} = 1.6759 \cong 2$$

$$x_3 = nmk = 84.6543 \Rightarrow k = \frac{84.6543}{nm} = \frac{84.6543}{53.5847} = 1.5798 \cong 2$$

$$n=32, \quad m=2 \quad \text{and} \quad k=2$$

After putting the values of  $x_1^*$ ,  $x_2^*$  and  $x_3^*$  in equation 4.23(i), we get the total cost as:

$$C = 10 \times 32 + 3 \times 54 + 1.5 \times 85 = 609.5$$

The feasibility of the solution is shown with the help of above example. Thus the requirement of sample for primary stage units is 32, the total of secondary stage units in each primary stage units  $nm = 54$  and the tertiary stage unit within each secondary stage units giving us total of  $nmk = 85$ , elementary units for the sample.

Now by using the LINGO (13) Software and primal-dual relationship theorem, the problem has been solved and the required optimal solution is obtained.

The final standard form of above problem as:

$$\left. \begin{array}{ll} \min C = 10x_1 + 3x_2 + 1.5x_3 & (i) \\ \text{subject to } \frac{14.6624}{X_1} + \frac{28.5466}{X_2} + \frac{29.0675}{X_3} \leq 1 & (ii) \\ \frac{16.8296}{X_1} + \frac{14.18}{X_2} + \frac{17.6945}{X_3} \leq 1 & (iii) \\ \frac{19.3601}{X_1} + \frac{5.3459}{X_2} + \frac{9.5403}{X_3} \leq 1 & (iv) \\ X_1, X_2, X_3 \geq 0 & \end{array} \right\} \quad (4.26)$$

The dual GPP of the above problem (4.26) is as follows:

$$\left. \begin{aligned} \max \quad v(w_{0i}^*) &= \left( (10/w_{01})^{w_{01}} \right) \times \left( (3/w_{02})^{w_{02}} \right) \times \left( (1.5/w_{03})^{w_{03}} \right) \\ &\times \left( (14.6624)^{w_{11}} \right) \times \left( (28.5466)^{w_{12}} \right) \times \left( (29.0675)^{w_{11}} \right) \times \left( (16.8296)^{w_{12}} \right) \times \\ &\times \left( (14.18)^{w_{12}} \right) \times \left( (17.6945)^{w_{11}} \right) \times \left( (19.3601)^{w_{12}} \right) \times \left( (5.3459)^{w_{11}} \right) \times \left( (9.5403)^{w_{12}} \right) \\ \text{subject to} \quad w_{01} + w_{02} + w_{03} &= 1 \quad (\text{normality condition}) \\ w_{01} - w_{11} - w_{21} - w_{31} &= 0 \\ w_{02} - w_{12} - w_{22} - w_{32} &= 0 \\ w_{03} - w_{13} - w_{32} - w_{33} &= 0 \end{aligned} \right\} \quad (\text{orthogonality condition})$$

$$w_{01}, w_{02}, w_{03}, w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_{33} \geq 0 \quad (\text{positivity condition})$$

Solving the above formulated dual problem, we have the corresponding solution as:

$$w_{01} = 0.4327892, \quad w_{02} = 0.3312079, \\ w_{03} = 0.2360029 \quad \text{and} \quad v(w_{0i}^*) = 782.1047$$

Using the primal-dual relationship theorem, we have the optimal solution of primal problem: i.e., the optimal sample sizes are computed as follows:

$$x_1^* = n = 34, x_2^* = nm = 86 \Rightarrow m = 3 \quad \text{and} \quad x_3^* = nmk = 123 \Rightarrow k = 1$$

Using the values of  $x_1^*$ ,  $x_2^*$  and  $x_3^*$ , we obtain the total cost as :

$$C = 10 \times 34 + 3 \times 86 + 1.5 \times 123 = 782.5$$

The feasibility of the solution is shown with the help of illustrated example. Thus the requirement of sample for primary stage units is 34, the total of secondary stage units in each primary stage units  $nm = 44$  and the tertiary stage unit within each secondary stage units giving us total of  $nmk = 123$ , elementary units for the sample.

#### 4.5 Conclusion

In this chapter an optimum allocation in three-stage sampling design with three variables are considered. Presently, the solution of mathematical programming problems, sampling problems, engineering problems, management problems etc. are very much dependent upon the efficiency of the software. With this regard, we have provided an effective method being handled manually and also by using LINGO software for obtaining optimum allocations in three-stage sampling with the help of primal-dual relationship of GP. The manual description of the solution procedure of GP is very simple in comparison to the complex analytical techniques used in

statistical literature. There may not be precise knowledge of parameters in the GP in real world due to insufficient information. The feasibility and effectiveness of the present approach has been illustrated by numerical example. GP optimization technique can be utilized in double sampling design having multiple characters; this is the wider application of the proposed approach.

## **CHAPTER – V**

### **Fuzzy Geometric Programming in Multivariate Stratified Sample Surveys in Presence of Non- Response with Quadratic Cost Function**

## CHAPTER – V

### FUZZY GEOMETRIC PROGRAMMING IN MULTIVARIATE STRATIFIED SAMPLE SURVEYS IN PRESENCE OF NON-RESPONSE WITH QUADRATIC COST FUNCTION

#### 5.1 Introduction

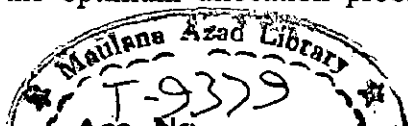
In sampling the precision of an estimator of the population parameters depend on the size of the sample and variability among the units of the population. If the population is heterogeneous and size of the sample depends on the cost of the survey, then it is likely to be impossible to get a sufficiently precise estimate with the help of simple random sampling from the entire population. In order to estimate the population mean or total with greater precision, the heterogeneous population is divided into mutually-exclusive, exhaustive and non-overlapping strata which will be more homogeneous than the entire population. The entire population is called Stratified Random Sampling. The problem of optimum allocation in stratified random sampling for univariate population is well known in sampling literature; see for example Cochran (1977) and Sukhatme *et al.* (1984). In multivariate stratified sample surveys problems the non-response can appear when the required data are not obtained. The problem of non-response may occur due to the refusal by respondents or they are not at home making the information of sample inaccessible. The problem of non-response occurs in almost all surveys. The extent of non- response depends on various factors such as type of the target population, type of the survey and the time of survey. For the problem of non-response in stratified sampling it may be assumed that every stratum is divided into two mutually exclusive and exhaustive groups of respondents and non respondents.

Hansen and Hurwitz (1946) presented a classical non-response theory which was first developed for the surveys in which the first attempt was made by mailing the questionnaires and a second attempt was made by personal interview to a sub sample of the non respondents. They constructed the estimator for the population mean and derived the expression for its variance and also worked out the optimum sampling fraction among the non respondents. El-Badry (1956) further extended the Hansen and Hurwitz's technique by sending waves of questionnaires to the non respondent units to increase the response rate. The generalized El-Badry's approach for different sampling design was given by Foradori (1961). Srinath (1971) suggested the selection

of sub samples by making several attempts. Khare (1987) investigated the problem of optimum allocation in stratified sampling in presence of non-response for fixed cost as well as for fixed precision of the estimate. Khan *et al.* (2008) suggested a technique for the problem of determining the optimum allocation and the optimum sizes of subsamples to various strata in multivariate stratified sampling in presence of non-response which is formulated as a nonlinear programming problem (NLPP). Varshney *et al.* (2011) formulated the multivariate stratified random sampling in the presence of non-response as a multi-objective integer nonlinear programming problem and a solution procedure is developed using lexicographic goal programming technique to determine the compromise allocation. Fatima and Ahsan (2011) address the problem of optimum allocation in stratified sampling in the presence of non-response and formulated as an all integer nonlinear programming problem (AINLPP). Varshney *et al.* (2012) has considered the multivariate stratified population with unknown strata weights and an optimum sampling design is proposed in the presence of non-response to estimate the unknown population means using DSS strategy and developed a solution procedure using goal programming technique and obtained an integer solution directly by the optimization software LINGO. Raghav *et al.* (2013) have discussed the various multi-objective optimization techniques in the multivariate stratified sample surveys in case of non-response.

Geometric programming (GP) is a smooth, systematic and an effective non-linear programming method used for solving the problems of sample surveys, management, transportation, engineering design etc. that takes the form of convex programming. The convex programming problems occurring in GP are generally represented by an exponential or power function. GP has certain advantages over the other optimization methods because it is usually much simpler to work with the dual than the primal one. The degree of difficulty (DD) plays a significant role for solving a non-linear programming problem by GP method.

Geometric programming (GP) has been known as an optimization tool for solving the problems in various fields. Duffin *et al.* (1967) and also Zener (1971) have discussed the basic concepts and theories of GP with application in engineering in their books. Beightler and Philips (1976) also published a famous book on GP and its application. Davis and Rudolph (1987) applied GP to optimal allocation of integrated samples in quality control. Ahmed and Charles (1987) applied geometric programming to obtain the optimum allocation problems in multivariate double sampling. Ojha and Das





(2010) have taken the multi-objective geometric programming problem being cost coefficient as continuous function with weighted mean and used the geometric programming technique for the solutions. Maqbool *et al.* (2011) and Shafiullah *et al.* (2013) have discussed the geometric programming approach to find the optimum allocations in multivariate two-stage sampling and three-stage sample surveys respectively.

In many real-world decision-making problems of sample surveys, environmental, social, economical and technical areas are of multiple-objectives. It is significant to realize that multiple objectives are often non-commensurable and in conflict with each other in optimization problems. The multi-objective models with fuzzy objectives are more realistic than deterministic of it. The concept of fuzzy set theory was firstly given by Zadeh (1965). Later on, Bellman and Zadeh (1970) used the fuzzy set theory to the decision-making problem. Tanaka *et al.* (1974) introduces the objective as fuzzy goal over the  $\alpha$ -cut of a fuzzy constraint set and Zimmermann (1978) gave the concept to solve multi-objective linear-programming problem. Biswal (1992) and Verma (1990) developed fuzzy geometric programming technique to solve multi-objective geometric programming (MOGP) problem. Islam (2005, 2010) has discussed modified geometric programming problem and its applications and also another fuzzy geometric programming technique to solve MOGPP and their applications. Fuzzy mathematical programming has been applied to several fields.

In this chapter, we have formulated the problem of non-response with significant travel cost where the cost is quadratic in  $\sqrt{n_h}$  in multivariate stratified sample surveys as a multi-objective geometric programming problem (MOGPP). The fuzzy programming approach has been described for solving the formulated MOGPP and the optimum allocations of sample sizes of respondents and non respondents are obtained. A numerical example is given to illustrate the procedure.

## 5.2 Formulation of the problem

In stratified sampling the population of  $N$  units is first divided into  $L$  non-overlapping subpopulation called strata, of sizes  $N_1, N_2, \dots, N_h, \dots, N_L$  with  $\sum_{h=1}^L N_h = N$  and the respective sample sizes within strata are drawn with independent simple random sampling denoted by  $n_1, n_2, \dots, n_h, \dots, n_L$  with  $\sum_{h=1}^L n_h = n$ .

Let for the  $h^{th}$  stratum:

$N_h$  : Stratum size.

$\bar{Y}_h$  : Stratum mean.

$S_h^2$  : Stratum variance.

$\hat{W}_{h1} = \hat{N}_{h1} / N$  : the estimated stratum weight among respondents.

$\hat{W}_{h2} = \hat{N}_{h2} / N$  : the estimated stratum weight among non-respondents.

$N_{h1}$  : the sizes of the respondents groups.

$N_{h2} = N_h - N_{h1}$  : the sizes of non respondents groups.

$n_h$  : Units are drawn from the  $h^{th}$  stratum. Further let out of  $n_h$ ,  $n_{h1}$  units belong to the respondents group.

$n_{h2} = n_h - n_{h1}$  : Units belong to the non respondents group.

$n = \sum_{h=1}^L n_h$  : The total sample size.

A more careful second attempt is made to obtain information on a random subsample of size  $r_h$  out of  $n_{h2}$  non respondents for the representation from the non respondents group of the sample.

$r_h = n_{h2} / k_h$ ;  $h = 1, 2, \dots, L$  : Subsamples of sizes at the second attempt to be drawn from  $n_{h2}$  non-respondent group of the  $h^{th}$  stratum. Where  $k_h \geq 1$  and  $1/k_h$  denote the sampling fraction among non-respondents. Since  $N_{h1}$  and  $N_{h2}$  are random variables hence their unbiased estimates are given as:

$\hat{N}_{h1} = n_{h1} N_h / n_h$  : The unbiased estimates of the respondents group.

$\hat{N}_{h2} = n_{h2} N_h / n_h$  : The unbiased estimate of the non respondents group.

$\bar{y}_{jh1}$ ;  $j = 1, \dots, p$  : denotes the sample means of  $j^{th}$  characteristic measured on the  $n_{h1}$  respondents at the first attempt.

$\bar{y}_{jh2(r_h)}$ ;  $j = 1, \dots, p$  : denotes the  $r_h$  sub sampled units from non respondents at the second attempt.

Using the estimator of Hansen and Hurwitz (1946), the stratum mean  $\bar{Y}_{jh}$  for  $j^{th}$  characteristic in the  $h^{th}$  stratum may be estimated by

$$\bar{y}_{jh(w)} = \frac{n_{h1} \bar{y}_{jh1} + n_{h2} \bar{y}_{jh2(r_h)}}{n_h} \quad (5.1)$$

It can be seen that  $\bar{y}_{jh(w)}$  is an unbiased estimate of the stratum mean  $\bar{Y}_{jh}$  of the  $h^{th}$  stratum for the  $j^{th}$  characteristic with a variance.

$$v(\bar{y}_{jh(w)}) = \left( \frac{1}{n_h} - \frac{1}{N_h} \right) S_{jh}^2 + \frac{W_{h2}^2 S_{jh2}^2}{r_h} - \frac{W_{h2}^2 S_{jh2}^2}{n_h} \quad (5.2)$$

where  $S_{jh}^2$  is the stratum variance of  $j^{th}$  characteristic in the  $h^{th}$  stratum;  $j=1,2,\dots,p$  and  $h=1,2,\dots,L$  given as:

$$S_{jh}^2 = \frac{1}{N_h - 1} \sum_{i=1}^{N_h} (y_{jhi} - \bar{Y}_{jh})^2$$

where  $y_{jhi}$  denote the value of the  $i^{th}$  unit of the  $h^{th}$  stratum for  $j^{th}$  characteristic.

$\bar{Y}_{jh} = 1/N_h \sum_{i=1}^{N_h} y_{jhi}$  : is the stratum mean of  $y_{jhi}$ .

$S_{jh2}^2$  is the stratum variance of the  $j^{th}$  characteristic in the  $h^{th}$  stratum among non respondents, given by:

$$\hat{S}_{jh2}^2 = \frac{1}{\hat{N}_{h2} - 1} \sum_{i=1}^{\hat{N}_{h2}} (y_{jhi} - \bar{Y}_{jh2})^2$$

$\bar{Y}_{jh2} = 1/\hat{N}_{h2} \sum_{i=1}^{\hat{N}_{h2}} y_{jhi}$  is the stratum mean of  $y_{jhi}$  among non respondents.

$\hat{W}_{h2} = \hat{N}_{h2}/N_h$  is stratum weight of non respondents in  $h^{th}$  stratum.

If the true values of  $S_{jh}^2$  and  $S_{jh2}^2$  are not known they can be estimated through a preliminary sample or the value of some previous occasion, if available, may be used.

Furthermore, the variance of  $\bar{y}_{j(w)} = \sum_{h=1}^L W_h \bar{y}_{jh(w)}$ , (ignoring fpc) is given as:

$$V(\bar{y}_{j(w)}) = \sum_{h=1}^L W_h^2 v(\bar{y}_{jh(w)})$$

$$V(\bar{y}_{j(w)}) = \sum_{h=1}^L \frac{W_h^2 (S_{jh}^2 - \hat{W}_{h2} \hat{S}_{jh2}^2)}{n_h} + \sum_{h=1}^L \frac{W_h^2 \hat{W}_{h2} \hat{S}_{jh2}^2}{r_h} = f_{0j}(n, r) \quad (5.3)$$

where  $\bar{y}_{j(w)}$  is an unbiased estimate of the overall population mean  $\bar{Y}_j$  of the  $j^{th}$  characteristic and  $V(\bar{y}_{jh(w)})$  is as given in Eqn.5.2.

$$\left. \begin{array}{l} \min V(\hat{\pi}_{sj}) = \sum_{h=1}^L \frac{a_{hj}}{n_h} + \sum_{h=1}^L \frac{a_{hj}}{r_h} \\ \text{subject to} \\ \sum_{h=1}^L (c_{h0} + c_{h1}W_{h1})n_h + \sum_{h=1}^L c_{h2}r_h + \sum_{h=1}^L t_{h0}\sqrt{n_h} + \sum_{h=1}^L t_{h2}\sqrt{r_h} \leq C_0 \\ n_h, r_h \geq 0 \text{ and } h=1,2,\dots,L \end{array} \right\}, j=1,2,\dots,p \quad (5.6)$$

Similarly, the expression (5.6) can be expressed in the standard primal GPP with cost function quadratic in  $\sqrt{n_h}$  where the travel cost is significant is given as follows:

$$\left. \begin{array}{l} \max f_{0j}(n, r) \\ \text{subject to } f_q(n, r) \leq 1 \\ n_h, r_h \geq 0, \quad h=1,2,\dots,L \end{array} \right\}, j=1,2,\dots,p \quad (5.7)$$

where  $f_q(n, r) = \sum_{i \in j[q]} d_i \{ (n_1^{p_{i1}} n_2^{p_{i2}} \dots n_L^{p_{iL}}) + (r_1^{p_{i1}} r_2^{p_{i2}} \dots r_L^{p_{iL}}) \}$ ,  $q=0,1,2,\dots,k$

or  $f_q(n, r) = \sum_{i \in j[q]} d_i \left\{ \left[ \prod_{h=1}^L n_h^{p_{ih}} \right] + \left[ \prod_{h=1}^L r_h^{p_{ih}} \right] \right\}$ ,  $d_i > 0$ ,  $n_h > 0$ ,  $q=0,1,2,\dots,k$ ,

$p_{ih}$ : arbitrary real numbers,  $d_i$ : positive and  $f_q(n)$ : posinomials

Let for simplicity  $f_{0j} = a_{hj}$  and  $d_i = a_{hj} = (c_{h0} + c_{h1}W_{h1})/C_0 = c_{h2}/C_0 = t_{h0}/C_0 = t_{h2}/C_0$

where  $f_q(n, r) = \sum_{h=1}^L \frac{(c_{h0} + c_{h1}W_{h1})}{C_0} n_h + \sum_{h=1}^L \frac{c_{h2}}{C_0} r_h + \sum_{h=1}^L \frac{t_{h0}}{C_0} \sqrt{n_h} + \sum_{h=1}^L \frac{t_{h2}}{C_0} \sqrt{r_h}$

The dual form of the primal GPP which is stated in (5.7) can be given as:

$$\left. \begin{array}{l} \max v_{0j}(w) = \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject to } \sum_{i \in [0]} w_i = 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i = 0 \quad (iii) \\ w_i \geq 0, q=0,1,\dots,k \text{ and } i=1,2,\dots,m_k \quad (iv) \end{array} \right\}, j=1,\dots,p. \quad (5.8)$$

The above formulated dual GPP (5.8) can be solved in the following two-steps:

**Step 1:** For the Optimum value of the objective function, the objective function always takes the form:

$$C_0(x^*) = \left( \frac{\text{Coeffi. of first term}}{w_{01}} \right)^{w_{01}} \times \left( \frac{\text{Coeffi. of second term}}{w_{02}} \right)^{w_{02}} \\ \times \dots \times \left( \frac{\text{Coeffi. of last term}}{w_{0k}} \right)^{w_{0k}} \left( \sum w' \text{ s in the first constraints} \right)^{\sum w' \text{ s in the first constraints}} \\ \left( \sum w' \text{ s in the last constraints} \right)^{\sum w' \text{ s in the last constraints}}$$

The multi-objective function for our problem is:

$$\prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i}$$

**Step 2:** The equations that can be used for GPP for the weights are given below:

$$\sum_{i \in j[o]} w_i \text{ in the objective function} = 1 \text{ (Normality condition)}$$

and for each primal variable  $n_h$  &  $\sqrt{n_h}$  and  $r_h$  &  $\sqrt{r_h}$  having m terms.

$$\sum_{i=1}^{m_k} (w_i \text{ for each term}) \times (\text{exponent on } n_h \text{ & } \sqrt{n_h} \text{ and } r_h \text{ & } \sqrt{r_h} \text{ in that term}) = 0 \\ \text{(Orthogonality condition)}$$

and  $w_i \geq 0$  (Positivity condition).

The above problem (5.8) has been solved with the help of steps (1-2) discussed above and the corresponding solutions  $w_{0i}^*$  is the unique solution to the dual constraint; it will also maximize the objective function for the dual problem. Next, the solution of the primal problem will be obtained using primal-dual relationship theorem which is given below:

**5.4 Primal-dual relationship theorem:** If  $w_{0i}^*$  is a maximizing point for dual problem

(5.8), each minimizing points  $(n_1, n_2, n_3, n_4 \text{ and } r_1, r_2, r_3, r_4)$  for primal problem

(5.7) satisfies the system of equations:

$$f_{0j}(n) = \begin{cases} w_{0i}^* v(w^*) & i \in J[0]_p \\ \frac{w_{ij}}{v_L(w_{0i}^*)}, & i \in J[L]_p \end{cases} \quad (5.9)$$

where  $L$  ranges over all positive integers for which  $v_L(w_{0i}^*) > 0$ .

Assuming a linear cost function the total cost  $C$  of the sample survey may be given as:

$$C = \sum_{h=1}^L c_{h0} n_h + \sum_{h=1}^L c_{h1} n_{h1} + \sum_{h=1}^L c_{h2} n_{h2}$$

where  $c_{h0}$  = the per unit cost of making the first attempt,

$c_{h1} = \sum_{j=1}^p c_{jh1}$  is the per unit cost for processing the results of all the  $p$  characteristics on the  $n_{h1}$  selected units from respondents group in the  $h^{th}$  stratum in the first attempt

and  $c_{h2} = \sum_{j=1}^p c_{jh2}$  :is the per unit cost for measuring and processing the results of all

the  $p$  characteristics on the  $r_h$  units selected from the non respondents group in the  $h^{th}$  stratum in the second attempt. Also,  $c_{jh1}$  and  $c_{jh2}$  are per unit costs of measuring the  $j^{th}$  characteristic in first and second attempts respectively. As  $n_{h1}$  is not known until the first attempt has been made, the quantity  $W_{h1} n_{h1}$  may be used as its expected value.

The total expected cost  $\hat{C}$  of the survey may be given as:

$$\hat{C} = \sum_{h=1}^L (c_{h0} + c_{h1} W_{h1}) n_h + \sum_{h=1}^L c_{h2} r_h + \sum_{h=1}^L t_{h0} \sqrt{n_h} + \sum_{h=1}^L t_{h2} \sqrt{r_h} \quad (5.4)$$

The problem therefore reduces to find the optimal values of sample sizes of respondents  $n_h^*$  and non-respondents  $r_h^*$  which are expressed as:

$$\left. \begin{aligned} \min (f_{0j}) &= \sum_{h=1}^L \frac{W_h^2 (S_{jh}^2 - W_{h2} S_{jh2}^2)}{n_h} + \sum_{h=1}^L \frac{W_h^2 W_{h2} S_{jh2}^2}{r_h} \\ \text{subject to} \\ \sum_{h=1}^L (c_{h0} + c_{h1} W_{h1}) n_h + \sum_{h=1}^L c_{h2} r_h + \sum_{h=1}^L t_{h0} \sqrt{n_h} + \sum_{h=1}^L t_{h2} \sqrt{r_h} &\leq C_0 \\ n_h, r_h &\geq 0 \text{ and } h = 1, 2, \dots, L \end{aligned} \right\}, j = 1, 2, \dots, p \quad (5.5)$$

### 5.3 Geometric Programming Formulation

The following multi-objective nonlinear programming problem (MONLPP) with the cost function quadratic in  $\sqrt{n_h}$  and significant travel cost is defined in equations (5.6) as follows

The optimal values of sample sizes of the respondents  $(n_h^*)$  and non-respondents  $(r_h^*)$  can be calculated with the help of the primal–dual relationship theorem (5.9).

### 5.5 Fuzzy Geometric Programming Approach

The solution procedure to solve the problem (5.15) consists of the following steps:

**Step-1:** Solve the MOGPP as a single objective problem using only one objective at a time and ignoring the others. These solutions are known as ideal solution.

**Step-2:** From the results of step-1, determine the corresponding values for every objective at each solution derived. Let  $(n^{(1)}, r^{(1)}), (n^{(2)}, r^{(2)}), \dots, (n^{(j)}, r^{(j)}), \dots, (n^{(p)}, r^{(p)})$  are the ideal solutions of the objective functions

$$f_{01}(n^{(1)}, r^{(1)}), f_{02}(n^{(2)}, r^{(2)}), \dots, f_{0j}(n^{(j)}, r^{(j)}), \dots, f_{0p}(n^{(p)}, r^{(p)}).$$

So  $U_j = \text{Max} \{f_{01}(n^{(1)}, r^{(1)}), f_{02}(n^{(2)}, r^{(2)}), \dots, f_{0p}(n^{(p)}, r^{(p)})\}$  and

$$L_j = f_{0j}^*(n^{(j)}, r^{(j)}), j=1,2,\dots,p.$$

$$\left[ \begin{array}{l} U_j \text{ and } L_j \text{ be the upper and lower bounds of the } j^{\text{th}} \text{ objective function} \\ f_{0j}(n, r), j=1,2,\dots,p. \end{array} \right]$$

**Step 3:** The membership function for the given problem can be define as:

$$\mu_j(f_{0j}(n, r)) = \begin{cases} 0, & \text{if } f_{0j}(n, r) \geq U_j \\ \frac{U_j(n, r) - f_{0j}(n, r)}{U_j(n, r) - L_j(n, r)}, & \text{if } L_j \leq f_{0j}(n, r) \leq U_j, j=1,2,\dots,p \\ 1, & \text{if } f_{0j}(n, r) \leq L_j \end{cases} \quad (5.10)$$

Here  $U_j(n, r)$  is a strictly monotonic decreasing function with respect to  $f_{0j}(n, r)$ .

The membership functions in Eqn. (5.10)

$$\text{i.e. } \mu_j(f_{0j}(n, r)), j=1, 2, \dots, p$$

Therefore the general aggregation function can be defined as

$$\mu_{\bar{D}}(n, r) = \mu_{\bar{D}} \{ \mu_1(f_{01}(n, r)), \mu_2(f_{02}(n, r)), \dots, \mu_p(f_{0p}(n, r)) \}$$

The fuzzy multi-objective formulation of the problem can be defined as:

$$\left. \begin{array}{l} \max \mu_{\bar{D}}(n, r) \\ \text{subject to} \\ \sum_{h=1}^L \frac{(c_{h0} + c_{h1}W_{h1})}{C_0} n_h + \sum_{h=1}^L \frac{c_{h2}}{C_0} r_h + \sum_{h=1}^L \frac{t_{h0}}{C_0} \sqrt{n_h} + \sum_{h=1}^L \frac{t_{h2}}{C_0} \sqrt{r_h} \leq 1; \\ n_h, r_h \geq 0 \text{ and } j=1,2,\dots,p. \end{array} \right\} \quad (5.11)$$

The problem to find the optimal values of  $(n^*, r^*)$  for this convex-fuzzy decision based on addition operator (like Tiwari *et. al.* (1987)). Therefore the problem (5.11) is reduced according to max-addition operator as:

$$\left. \begin{aligned} \max \mu_D(n^*, r^*) &= \sum_{j=1}^p \mu_j(f_{0j}(n, r)) = \sum_{j=1}^p \frac{U_j - (f_{0j}(n, r))}{U_j - L_j} \\ \text{subject to } f_q(n, r) &\leq 1; \\ 0 &\leq \mu_j(f_{0j}(n, r)) \leq 1 \\ n_h, r_h &\geq 0 \text{ and } j=1, 2, \dots, p. \end{aligned} \right\} \quad (5.12)$$

The above problem (5.12) reduces to

$$\left. \begin{aligned} \max \mu_D(n^*, r^*) &= \sum_{j=1}^p \left\{ \frac{U_j}{U_j - L_j} - \frac{(f_{0j}(n, r))}{U_j - L_j} \right\} \\ \text{subject to } f_q(n, r) &\leq 1; \\ n_h, r_h &\geq 0 \text{ and } j=1, 2, \dots, p. \end{aligned} \right\} \quad (5.13)$$

The problem (5.13) maximizes if the function  $F_{0j}(n) = \left\{ \frac{(f_{0j}(n, r))}{U_j - L_j} \right\}$  attain the minimum values. Therefore the problem (5.13) reduces into the primal problem (5.14) define as:

$$\left. \begin{aligned} \min \sum_{j=0}^p F_{0j}(n, r) \\ \text{subject to } f_q(n, r) &\leq 1; \\ n_h, r_h &\geq 0 \text{ and } h=1, 2, \dots, L \end{aligned} \right\} \quad (5.14)$$

The dual form of the Primal GPP which is stated in (5.14) can be given as:

$$\left. \begin{aligned} \max v(w) &= \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject } \sum_{i \in [0]} w_i &= 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i &= 0 \quad (iii) \\ w_i &\geq 0, q=0, 1, \dots, k \text{ and } i=1, 2, \dots, m_k \quad (iv) \end{aligned} \right\} \quad (5.15)$$



The optimal values of sample sizes of the respondents ( $n_h^*$ ) and non-respondents ( $r_h^*$ ) can be calculated with the help of the primal–dual relationship theorem (5.9).

## 5.6 Numerical

In the following table the stratum sizes, stratum weights, stratum standard deviations, measurement costs and the travel costs within the stratum are given for two characteristics under study in a population stratified in four strata. The data are mainly from Khan *et al.* (2008). The travelling costs  $t_{h0}$  and  $t_{h2}$  are assumed.

The total budget available for the survey is taken as  $C_0 = 5000$ . The relative values of the variances of the non-respondents and respondents, that is  $S_{jh2}^2 / S_{jh}^2$  is assumed to be constant and equal to 0.25 for  $j = 1, 2$  and  $h = 1, 2, 3, 4$ . However, these ratios may vary from stratum to stratum and from characteristic to characteristic and can be handled accordingly.

**Table 1: Data for four Strata and two characteristics**

$h$	$N_h$	$S_{1h}^2$	$S_{2h}^2$	$w_{h1}$	$w_{h2}$	$c_{h0}$	$c_{h1}$	$c_{h2}$	$t_{h2}$	$t_{h0}$
1	1214	4817.72	8121.15	0.7	0.30	1	2	3	0.5	2
2	822	6251.26	7613.52	0.80	0.20	1	3	4	0.5	2.5
3	1028	3066.16	1456.4	0.75	0.25	1	4	5	0.5	3
4	786	6207.25	6977.72	0.72	0.28	1	5	6	0.5	4.5

For solving MOGPP by using fuzzy programming, we shall first solve the two sub-problems:

**Sub problem1:** On substituting the table values in sub-problem 1, we have obtained the expressions given below:

$$\begin{aligned}
\min f_{01} = & \frac{456.3344}{n_1} + \frac{261.8965}{n_2} + \frac{209.5529}{n_3} + \frac{230.9097}{n_4} + \\
& \frac{11.10002688}{r_1} + \frac{2.75680566}{r_2} + \frac{3.492547875}{r_3} + \frac{4.866484}{r_4} + \\
\text{subject to } & 0.00048n_1 + 0.00068n_2 + 0.0008n_3 + 0.00092n_4 + \\
& 0.0006r_1 + 0.0008r_2 + 0.001r_3 + 0.0012r_4 + \\
& 0.0001\sqrt{n_1} + 0.0001\sqrt{n_2} + 0.0001\sqrt{n_3} + 0.0001\sqrt{n_4} + \\
& 0.0004\sqrt{r_1} + 0.0005\sqrt{r_2} + 0.0006\sqrt{r_3} + 0.0009\sqrt{r_4} \leq 1 \\
& n_h \geq 0, r_h \geq 0, n_h, r_h \text{ are integers; } h=1,2,\dots,L
\end{aligned} \tag{5.16}$$

The dual of the above problem (5.16) is obtained as:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left( (456.3344/w_{01})^{w_{01}} \times (261.8965/w_{02})^{w_{02}} \times (209.5529/w_{03})^{w_{03}} \right. \\
 & \times (230.9097/w_{04})^{w_{04}} \times (11.100027/w_{05})^{w_{05}} \times (2.756806/w_{06})^{w_{06}} \\
 & \times (3.492548/w_{07})^{w_{07}} \times (4.866484/w_{08})^{w_{08}} \times \left( \frac{0.00048}{w_{11}} \right)^{w_{11}} \\
 & \times \left( \frac{0.00068}{w_{12}} \right)^{w_{12}} \times \left( \frac{0.0008}{w_{13}} \right)^{w_{13}} \times \left( \frac{0.00092}{w_{14}} \right)^{w_{14}} \\
 & \times \left( \frac{0.0006}{w_{15}} \right)^{w_{15}} \times \left( \frac{0.0008}{w_{16}} \right)^{w_{16}} \times \left( \frac{0.001}{w_{17}} \right)^{w_{17}} \times \left( \frac{0.0012}{w_{18}} \right)^{w_{18}} \\
 & \times \left( \frac{0.0001}{w_{19}} \right)^{w_{19}} \times \left( \frac{0.0001}{w_{20}} \right)^{w_{20}} \times \left( \frac{0.0001}{w_{21}} \right)^{w_{21}} \times \left( \frac{0.0001}{w_{22}} \right)^{w_{22}} \\
 & \times \left( \frac{0.0004}{w_{23}} \right)^{w_{23}} \times \left( \frac{0.0005}{w_{24}} \right)^{w_{24}} \times \left( \frac{0.0006}{w_{25}} \right)^{w_{25}} \times \left( \frac{0.0009}{w_{26}} \right)^{w_{26}} \\
 & \times ((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} \\
 & + w_{19} + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})^{\wedge} \\
 & (w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} + w_{19} \\
 & + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})); \quad (i) \\
 \text{subject to} \quad & w_{01} + w_{02} + w_{03} + w_{04} + w_{05} + w_{06} + w_{07} + w_{08} = 1; \text{ (normality condition)} \quad (ii) \\
 & \left. \begin{aligned}
 -w_{01} + w_{11} + (1/2)w_{19} &= 0 \\
 -w_{02} + w_{12} + (1/2)w_{20} &= 0 \\
 -w_{03} + w_{13} + (1/2)w_{21} &= 0 \\
 -w_{04} + w_{14} + (1/2)w_{22} &= 0 \\
 -w_{05} + w_{15} + (1/2)w_{23} &= 0 \\
 -w_{06} + w_{16} + (1/2)w_{24} &= 0 \\
 -w_{07} + w_{17} + (1/2)w_{25} &= 0 \\
 -w_{08} + w_{18} + (1/2)w_{26} &= 0
 \end{aligned} \right\} \quad \text{(orthogonality condition)} \quad (iii) \\
 & \left. \begin{aligned}
 w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{06}, w_{07}, w_{08} &> 0; \\
 w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, w_{20} &> 0 \\
 w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26} &\geq 0
 \end{aligned} \right\} \quad \text{(positivity condition)} \quad (iv)
 \end{aligned} \tag{5.17}$$

For orthogonality condition defined in expression 5.17(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
w_{01} \\
w_{02} \\
w_{03} \\
w_{04} \\
w_{05} \\
w_{06} \\
w_{07} \\
w_{08} \\
w_{11} \\
w_{12} \\
w_{13} \\
w_{14} \\
w_{15} \\
w_{16} \\
w_{17} \\
w_{18} \\
w_{19} \\
w_{20} \\
w_{21} \\
w_{22} \\
w_{23} \\
w_{24} \\
w_{25} \\
w_{26}
\end{pmatrix}$$

$$\begin{aligned}
-w_{01} + w_{11} + (1/2)w_{19} &= 0 \\
-w_{02} + w_{12} + (1/2)w_{20} &= 0 \\
-w_{03} + w_{13} + (1/2)w_{21} &= 0 \\
-w_{04} + w_{14} + (1/2)w_{22} &= 0 \\
-w_{05} + w_{15} + (1/2)w_{23} &= 0 \\
-w_{06} + w_{16} + (1/2)w_{24} &= 0 \\
-w_{07} + w_{17} + (1/2)w_{25} &= 0 \\
-w_{08} + w_{18} + (1/2)w_{26} &= 0
\end{aligned}$$

Solving the above formulated dual problem (5.17) with the help of LINGO software, we have the corresponding dual solutions as follows:

$$\begin{aligned}
w_{01} &= 0.2306558, w_{02} = 0.2079228, w_{03} = 0.2017047, w_{04} = 0.2270049, w_{05} = 0.04094539, \\
w_{06} &= 0.02376402, w_{07} = 0.02986826, w_{08} = 0.03813407, \text{ and } v(w^*) = 4.175039.
\end{aligned}$$

Using the primal-dual-relationship theorem (5.9), we have the optimal solution of primal problem; *i.e.*, the optimal sample sizes of respondents and non respondents are computed as follows:

$$f_{0j}(n, r) = w_{0i}^* v(w_{0i}^*)$$

In expression (5.16), we first keep the  $r$  constant and calculate the values of  $n$  as:

$$f_{01}(n_1, r) = w_{01}^* v(w_{01}^*) \Rightarrow n_1 \cong 474 \quad f_{02}(n_2, r) = w_{02}^* v(w_{02}^*) \Rightarrow n_2 \cong 302$$

$$f_{03}(n_3, r) = w_{03}^* v(w_{03}^*) \Rightarrow n_3 \cong 249 \quad f_{04}(n_4, r) = w_{04}^* v(w_{04}^*) \Rightarrow n_4 \cong 244$$

Now, from the expression (16), we keep the  $n$  constant and calculate the values of  $r$  as:

$$f_{01}(n, r_1) = w_{01}^* v(w_{01}^*) \Rightarrow r_1 \cong 65 \quad f_{02}(n, r_2) = w_{02}^* v(w_{02}^*) \Rightarrow r_2 \cong 28$$

$$f_{03}(n, r_3) = w_{03}^* v(w_{03}^*) \Rightarrow r_3 \cong 28 \quad f_{04}(n, r_4) = w_{04}^* v(w_{04}^*) \Rightarrow r_4 \cong 31$$

The optimal values and the objective function value are given below:

$$n_1^* = 474, n_2^* = 302, n_3^* = 249 \text{ and } n_4^* = 244;$$

$$r_1^* = 65, r_2^* = 28, r_3^* = 28 \text{ and } r_4^* = 31 \text{ and the objective value of the primal problem is } 4.175039.$$

**Sub problem 2:** On substituting the table values in sub-problem 2, we have obtained the expressions given below:

$$\left. \begin{aligned} \text{Min } f_{02} = & \frac{769.2353}{n_1} + \frac{318.9684}{n_2} + \frac{99.53584}{n_3} + \frac{259.5712}{n_4} + \\ & \frac{18.7111296}{r_1} + \frac{3.35756232}{r_2} + \frac{1.658930625}{r_3} + \frac{5.47053248}{r_4}; \\ \text{Subject to} & \\ & 0.00048n_1 + 0.00068n_2 + 0.0008n_3 + 0.00092n_4 \\ & + 0.0006r_1 + 0.0008r_2 + 0.001r_3 + 0.0012r_4 \\ & + 0.0001\sqrt{n_1} + 0.0001\sqrt{n_2} + 0.0001\sqrt{n_3} + 0.0001\sqrt{n_4} \\ & + 0.0004\sqrt{r_1} + 0.0005\sqrt{r_2} + 0.0006\sqrt{r_3} + 0.0009\sqrt{r_4} \leq 1 \\ & n_h \geq 0, r_h \geq 0, n_h, r_h \text{ are integers; } h=1,2,\dots,L \end{aligned} \right\} \quad (5.18)$$

The dual of the above problem (5.18) is obtained as follows:

$$\begin{aligned}
\max v(w_{0i}^*) = & \left( (769.2353/w_{01})^{w_{01}} \times (318.9684/w_{02})^{w_{02}} \times (99.53584/w_{03})^{w_{03}} \right. \\
& \times (259.5712/w_{04})^{w_{04}} \times (18.7111296/w_{05})^{w_{05}} \times (3.35756232/w_{06})^{w_{06}} \\
& \times (1.658930625/w_{07})^{w_{07}} \times (5.47053248/w_{08})^{w_{08}} \times \left( \frac{0.00048}{w_{11}} \right)^{w_{11}} \\
& \times \left( \frac{0.00068}{w_{12}} \right)^{w_{12}} \times \left( \frac{0.0008}{w_{13}} \right)^{w_{13}} \times \left( \frac{0.00092}{w_{14}} \right)^{w_{14}} \times \left( \frac{0.0006}{w_{15}} \right)^{w_{15}} \\
& \times \left( \frac{0.0008}{w_{16}} \right)^{w_{16}} \times \left( \frac{0.001}{w_{17}} \right)^{w_{17}} \times \left( \frac{0.0012}{w_{18}} \right)^{w_{18}} \times \left( \frac{0.0001}{w_{19}} \right)^{w_{19}} \\
& \times \left( \frac{0.0001}{w_{20}} \right)^{w_{20}} \times \left( \frac{0.0001}{w_{21}} \right)^{w_{21}} \times \left( \frac{0.0001}{w_{22}} \right)^{w_{22}} \times \left( \frac{0.0004}{w_{23}} \right)^{w_{23}} \\
& \times \left( \frac{0.0005}{w_{24}} \right)^{w_{24}} \times \left( \frac{0.0006}{w_{25}} \right)^{w_{25}} \times \left( \frac{0.0009}{w_{26}} \right)^{w_{26}} \times \\
& ((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} + \\
& w_{19} + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})^{\wedge} \\
& (w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} \\
& + w_{19} + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})); \quad (i) \\
\text{subject to } & w_{01} + w_{02} + w_{03} + w_{04} + w_{05} + w_{06} + w_{07} + w_{08} = 1; \text{ (normality condition) } (ii) \\
& \left. \begin{aligned}
& -w_{01} + w_{11} + (1/2)w_{19} = 0 \\
& -w_{02} + w_{12} + (1/2)w_{20} = 0 \\
& -w_{03} + w_{13} + (1/2)w_{21} = 0 \\
& -w_{04} + w_{14} + (1/2)w_{22} = 0 \\
& -w_{05} + w_{15} + (1/2)w_{23} = 0 \\
& -w_{06} + w_{16} + (1/2)w_{24} = 0 \\
& -w_{07} + w_{17} + (1/2)w_{25} = 0 \\
& -w_{08} + w_{18} + (1/2)w_{26} = 0
\end{aligned} \right\} \text{ (orthogonality condition) } (iii) \\
& \left. \begin{aligned}
& w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{06}, w_{07}, w_{08} > 0; \\
& w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, \\
& w_{20}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26} \geq 0
\end{aligned} \right\} \text{ (positivity condition) } (iv)
\end{aligned}
\tag{5.19}$$

For orthogonality condition defined in expression 5.19(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
w_{01} \\
w_{02} \\
w_{03} \\
w_{04} \\
w_{05} \\
w_{06} \\
w_{07} \\
w_{08} \\
w_{11} \\
w_{12} \\
w_{13} \\
w_{14} \\
w_{15} \\
w_{16} \\
w_{17} \\
w_{18} \\
w_{19} \\
w_{20} \\
w_{21} \\
w_{22} \\
w_{23} \\
w_{24} \\
w_{25} \\
w_{26}
\end{pmatrix}$$

$$-w_{01} + w_{11} + (1/2)w_{19} = 0$$

$$-w_{02} + w_{12} + (1/2)w_{20} = 0$$

$$-w_{03} + w_{13} + (1/2)w_{21} = 0$$

$$-w_{04} + w_{14} + (1/2)w_{22} = 0$$

$$-w_{05} + w_{15} + (1/2)w_{23} = 0$$

$$-w_{06} + w_{16} + (1/2)w_{24} = 0$$

$$-w_{07} + w_{17} + (1/2)w_{25} = 0$$

$$-w_{08} + w_{18} + (1/2)w_{26} = 0$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$\begin{aligned}
w_{01} &= 0.2854095, w_{02} = 0.2187303, w_{03} = 0.1325806, w_{04} = 0.2294348, w_{05} = 0.04989059, \\
w_{06} &= 0.05057448, w_{07} = 0.02498275, w_{08} = 0.01974803, \text{ and } v(w^*) = 4.593918.
\end{aligned}$$

The optimal values of sample sizes of respondents and non-respondents  $(n_h^*, r_h^*)$  can be calculated with the help of the primal–dual relationship theorem (5.9) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 587, n_2^* = 317, n_3^* = 163 \text{ and } n_4^* = 246;$$

$$r_1^* = 81, r_2^* = 29, r_3^* = 18 \text{ and } r_4^* = 31 \text{ and the objective value of the primal problem is } 4.593918.$$

Now the pay-off matrix of the above problems is given below:

$$\begin{array}{cc} & f_{01}(n, r) & f_{02}(n, r) \\ \begin{array}{c} (n^{(1)}, r^{(1)}) \\ (n^{(2)}, r^{(2)}) \end{array} & \begin{bmatrix} 4.175039 & 4.793168 \\ 4.403294 & 4.593918 \end{bmatrix} \end{array}$$

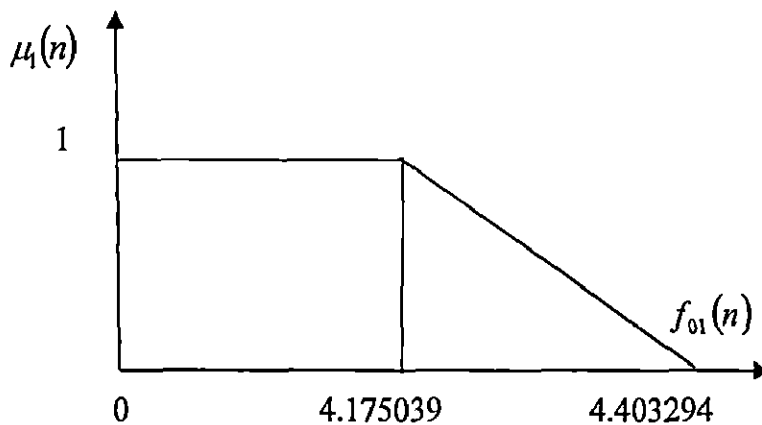
The lower and upper bond of  $f_{01}(n, r)$  and  $f_{02}(n, r)$  can be obtained from the pay-off matrix

$$4.175039 \leq f_{01}(n, r) \leq 4.403294 \text{ and } 4.593918 \leq f_{02}(n, r) \leq 4.793168.$$

Let  $\mu_1(n, r)$  and  $\mu_2(n, r)$  be the fuzzy membership function of the objective function  $f_{01}(n, r)$  and  $f_{02}(n, r)$  respectively and they are defined as:

$$\mu_1(n, r) = \begin{cases} 1 & , \text{if } f_{01}(n, r) \leq 4.175039 \\ \frac{4.403294 - f_{01}(n, r)}{0.228255} & , \text{if } 4.175039 \leq f_{01}(n, r) \leq 4.403294 \\ 0 & , \text{if } f_{01}(n, r) \geq 4.403294 \end{cases}$$

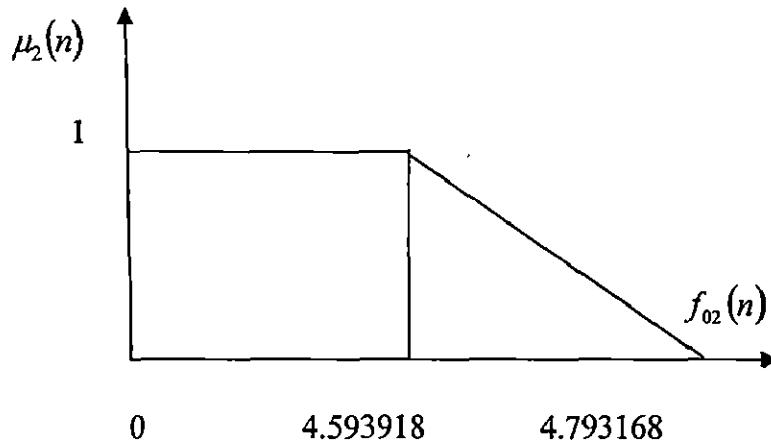
The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_2(n, r) = \begin{cases} 1 & , \text{if } Z_2(n, r) \leq 4.593918 \\ \frac{4.793168 - Z_2(n, r)}{0.19925} & , \text{if } 4.593918 \leq Z_2(n, r) \leq 4.793168 \\ 0 & , \text{if } Z_2(n, r) \geq 4.793168 \end{cases}$$



The following figure illustrated the graph of the fuzzy membership function  $\mu_2(n)$



On applying the max-addition operator, the MOGPP, the standard primal problem reduces to the problem as:

$$\begin{aligned} \max \quad & \left\{ \begin{aligned} & (\mu_1(n,r) + \mu_2(n,r)) \\ & \left\{ \frac{4.403294 - f_{02}(n,r)}{0.228255} + \frac{4.793168 - f_{02}(n,r)}{0.19925} \right\} \\ & \left\{ 43.3472 - \left( \frac{f_{01}(n,r)}{0.228255} + \frac{f_{02}(n,r)}{0.19925} \right) \right\} \end{aligned} \right\} \\ \text{subject to} \quad & 0.00048n_1 + 0.00068n_2 + 0.0008n_3 + 0.00092n_4 \\ & + 0.0006r_1 + 0.0008r_2 + 0.001r_3 + 0.0012r_4 \\ & + 0.0001\sqrt{n_1} + 0.0001\sqrt{n_2} + 0.0001\sqrt{n_3} + 0.0001\sqrt{n_4} \\ & + 0.0004\sqrt{r_1} + 0.0005\sqrt{r_2} + 0.0006\sqrt{r_3} + 0.0009\sqrt{r_4} \leq 1 \\ & n_h \geq 0, r_h \geq 0; h=1,2,\dots,L \end{aligned} \quad (5.20)$$

In order to maximize the above problem, we have to minimize

$$\left( \frac{f_{01}(n,r)}{0.228255} + \frac{f_{02}(n,r)}{0.19925} \right), \text{ as described below:}$$

Hence the dual problem of the above final formulated problem (5.22) is given as:

Degree of Difficulty of the problem (5.22) is  $= (24 - (8 + 1)) = 15$ .

$$(5.22) \quad \left\{ \begin{array}{l} \min \left\{ \frac{n_1}{5859.8847} + \frac{n_2}{2748.2334} + \frac{n_3}{1417.6227} + \frac{n_4}{2314.3744} + \frac{r_1}{142.5377} + \frac{r_2}{28.9287} + \frac{r_3}{23.627} + \frac{r_4}{48.7761} \right\} \\ \text{subject to } 0.00048n_1 + 0.00068n_2 + 0.0008n_3 + 0.00092n_4 \\ + 0.0006r_1 + 0.0008r_2 + 0.001r_3 + 0.0012r_4 \\ + 0.0001\sqrt{n_1} + 0.0001\sqrt{n_2} + 0.0001\sqrt{n_3} + 0.0001\sqrt{n_4} \\ + 0.0004\sqrt{r_1} + 0.0005\sqrt{r_2} + 0.0006\sqrt{r_3} + 0.0009\sqrt{r_4} \leq 1 \\ n_h \geq 0, r_h \geq 0; h = 1, 2, \dots, L \end{array} \right.$$

quadratic in  $\sqrt{n_h}$  and significant travel cost can be as follows:

On applying the max-addition operator, the final primal MOGPP with cost function

$$(5.21) \quad \left\{ \begin{array}{l} \min \left\{ \begin{array}{l} (4.381065) \times \left( \frac{n_1}{4563344} + \frac{n_2}{2618965} + \frac{n_3}{209.5529} + \frac{n_4}{230.9097} + \frac{r_1}{1110002688} + \frac{r_2}{2.75680566} + \frac{r_3}{3.492547875} + \frac{r_4}{4.866484} \right) \\ (5.018821) \times \left( \frac{n_1}{769.2353} + \frac{n_2}{318.9684} + \frac{n_3}{99.53584} + \frac{n_4}{259.5712} + \frac{r_1}{18.7111296} + \frac{r_2}{3.35756232} + \frac{r_3}{1.658930625} + \frac{r_4}{5.47053248} \right) \end{array} \right\} \\ \text{subject to } 0.00048n_1 + 0.00068n_2 + 0.0008n_3 + 0.00092n_4 \\ + 0.0006r_1 + 0.0008r_2 + 0.001r_3 + 0.0012r_4 \\ + 0.0001\sqrt{n_1} + 0.0001\sqrt{n_2} + 0.0001\sqrt{n_3} + 0.0001\sqrt{n_4} \\ + 0.0004\sqrt{r_1} + 0.0005\sqrt{r_2} + 0.0006\sqrt{r_3} + 0.0009\sqrt{r_4} \leq 1 \\ n_h \geq 0, r_h \geq 0; h = 1, 2, \dots, L \end{array} \right.$$

$$\begin{aligned}
\max v(w_{0i}^*) = & \left( (5859.8847/w_{01})^{w_{01}} \right) \times \left( (2748.2334/w_{02})^{w_{02}} \right) \times \left( (1417.6227/w_{03})^{w_{03}} \right) \\
& \times \left( (2314.3744/w_{04})^{w_{04}} \right) \times \left( (48.7761/w_{08})^{w_{08}} \right) \times \left( (142.5377/w_{05})^{w_{05}} \right) \\
& \times \left( (28.9287/w_{06})^{w_{06}} \right) \times \left( (23.627/w_{07})^{w_{07}} \right) \times \left( \frac{0.00048}{w_{11}} \right)^{w_{11}} \\
& \times \left( \frac{0.00068}{w_{12}} \right)^{w_{12}} \times \left( \frac{0.0008}{w_{13}} \right)^{w_{13}} \times \left( \frac{0.00092}{w_{14}} \right)^{w_{14}} \\
& \times \left( \frac{0.0006}{w_{15}} \right)^{w_{15}} \times \left( \frac{0.0008}{w_{16}} \right)^{w_{16}} \times \left( \frac{0.001}{w_{17}} \right)^{w_{17}} \\
& \times \left( \frac{0.0012}{w_{18}} \right)^{w_{18}} \times \left( \frac{0.0001}{w_{19}} \right)^{w_{19}} \times \left( \frac{0.0001}{w_{20}} \right)^{w_{20}} \\
& \times \left( \frac{0.0001}{w_{21}} \right)^{w_{21}} \times \left( \frac{0.0001}{w_{22}} \right)^{w_{22}} \times \left( \frac{0.0004}{w_{23}} \right)^{w_{23}} \\
& \times \left( \frac{0.0005}{w_{24}} \right)^{w_{24}} \times \left( \frac{0.0006}{w_{25}} \right)^{w_{25}} \times \left( \frac{0.0009}{w_{26}} \right)^{w_{26}} \times \\
& ((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} + \\
& w_{19} + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})^\wedge \\
& (w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18} + \\
& w_{19} + w_{20} + w_{21} + w_{22} + w_{23} + w_{24} + w_{25} + w_{26})); (i) \\
\text{subject to } & w_{01} + w_{02} + w_{03} + w_{04} + w_{05} + w_{06} + w_{07} + w_{08} = 1; \text{ (normality condition) } (ii) \\
& \left. \begin{aligned}
& -w_{01} + w_{11} + (1/2)w_{19} = 0 \\
& -w_{02} + w_{12} + (1/2)w_{20} = 0 \\
& -w_{03} + w_{13} + (1/2)w_{21} = 0 \\
& -w_{04} + w_{14} + (1/2)w_{22} = 0 \\
& -w_{05} + w_{15} + (1/2)w_{23} = 0 \\
& -w_{06} + w_{16} + (1/2)w_{24} = 0 \\
& -w_{07} + w_{17} + (1/2)w_{25} = 0 \\
& -w_{08} + w_{18} + (1/2)w_{26} = 0
\end{aligned} \right\} \text{ (orthogonality condition) } (iii) \\
& \left. \begin{aligned}
& w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{06}, w_{07}, w_{08} > 0; \\
& w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18}, w_{19}, \\
& w_{20}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26} \geq 0
\end{aligned} \right\} \text{ (positivity condition) } (iv)
\end{aligned}
\tag{5.23}$$

For orthogonality condition defined in expression 5.23(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & (1/2)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
w_{01} \\
w_{02} \\
w_{03} \\
w_{04} \\
w_{05} \\
w_{06} \\
w_{07} \\
w_{08} \\
w_{11} \\
w_{12} \\
w_{13} \\
w_{14} \\
w_{15} \\
w_{16} \\
w_{17} \\
w_{18} \\
w_{19} \\
w_{20} \\
w_{21} \\
w_{22} \\
w_{23} \\
w_{24} \\
w_{25} \\
w_{26}
\end{pmatrix}$$

$$\begin{aligned}
-w_{01} + w_{11} + (1/2)w_{19} &= 0 \\
-w_{02} + w_{12} + (1/2)w_{20} &= 0 \\
-w_{03} + w_{13} + (1/2)w_{21} &= 0 \\
-w_{04} + w_{14} + (1/2)w_{22} &= 0 \\
-w_{05} + w_{15} + (1/2)w_{23} &= 0 \\
-w_{06} + w_{16} + (1/2)w_{24} &= 0 \\
-w_{07} + w_{17} + (1/2)w_{25} &= 0 \\
-w_{08} + w_{18} + (1/2)w_{26} &= 0
\end{aligned}$$

After solving the formulated dual problem (5.23) using LINGO software we obtain the following values of the dual variables which are given as:

$$w_{01}=0.2610739, w_{02}=0.2127708, w_{03}=0.1657664, w_{04}=0.2270332, w_{05}=0.04629584, \\
w_{06}=0.02431073, w_{07}=0.02461037, w_{08}=0.03813880, \text{ and } v(w^*)=41.83433.$$

The optimal values of sample sizes of respondents and non-respondents  $(n_h^*, r_h^*)$  can be calculated with the help of the primal–dual relationship theorem (5.9) as we have calculated in the sub-problem 1 are given as follows:

$n_1^* = 537, n_2^* = 309, n_3^* = 204$  and  $n_4^* = 244$ ;  $r_1^* = 74, r_2^* = 28, r_3^* = 23$  and  $r_4^* = 31$  and the objective value of the primal problem is 41.83433.

## 5.7 Conclusion

This chapter provides the use of fuzzy programming for solving a multi-objective geometric programming problem (MOGPP). The problem of non-response with significant travel costs where the cost is quadratic in  $\sqrt{n_h}$  in multivariate stratified sample surveys is formulated as an MOGPP. The fuzzy programming approach is described for solving the formulated MOGPP. The formulated MOGPP is solved with the help of LINGO software and the dual solution is obtained. The optimum allocation of sample sizes of respondents and non-respondents are obtained with the help of dual solution and primal-dual relationship theorem. An illustrative numerical example is given to ascertain the practical utility of the proposed method in sample survey problems in the presence of non-response.

## **CHAPTER – VI**

### **Fuzzy Geometric Programming Approach in Multivariate Stratified Sample Surveys Under Two Stages Randomized Model Response**

## CHAPTER – VI

### **FUZZY GEOMETRIC PROGRAMMING APPROACH IN MULTIVARIATE STRATIFIED SAMPLE SURVEYS UNDER TWO STAGES RANDOMIZED RESPONSE MODEL**

#### **6.1 Introduction**

The questionnaires of the social survey generally contain questions on sensitive topics such as habitual tax evasion, drunken driving, drug addiction, sexual behavior, family income etc. In such situations, interviewees will be unwilling to give truthful answers, while interviewers will be uncomfortable when enquiring about the very personal situations. In order to reduce non-response, response bias and to promote respondent co-operation, improve upon the accuracy levels, a survey technique different from open or direct survey was needed that made people comfortable and encouraged to give truthful and faithful answers. Warner (1965) introduced a technique of generating 'randomized response' (RR) as a device to protect a respondent's privacy and secrecy for reducing the rate of non-response. The Warner's model requires the interviewee to give a "Yes" or "No" answer either to a sensitive question or to its negative, depending on the outcome of a randomizing device not disclosed to the interviewer.

Mangat and Singh (1990) proposed a two-stage RR model based on Warner's model. Further in (1994), Mangat proposed another RR model which has the advantages of simplicity over that of Mangat and Singh (1990) model. Hong *et al.* (1994) suggested a stratified RR technique that applied the same randomization device to every stratum. The applicability of these models are discussed in Singh and Mangat (1996). Under Hong *et al.* (1994) proportional sampling assumption, it may be easy to derive the variance of the proposed estimator. However, it may cause a high cost in terms of time, effort and money because of the difficulty in obtaining a proportional sample from some stratum. To overcome the above problem, we have considered the stratified randomized response technique under optimal allocation, which is more competent than proportional. Many authors have worked on RR Techniques that either improve upon Warner's technique or provide alternative procedures for more complicated situations and others, some of them are Fox and Tracy (1986), Chaudhuri and Mukerjee (1988), Chaudhuri (2001), Padmawar and Vijayan (2000), Tracy and

## 6.2 Formulation of the problem for two-stage Randomized Response Model

Under two-stage Randomized Response model at stage 1 an individual respondent selected in the sample from  $h^{th}$  stratum of a stratified population is introduced to use the randomization device  $R_{1h}$  which consists of the following two statements:

- (i) "I belong to the sensitive group" and
- (ii) "Go to the randomization device  $R_{2h}$  at the second stage" with known probabilities  $M_h$  and  $(1 - M_h)$  of (i) and (ii) respectively.

At stage 2 the respondents are instructed to use the randomization device  $R_{2h}$  which consists of the following two statements:

- (i) "I belong to the sensitive group" and
- (ii) "I do not belong to the sensitive group" with known probabilities  $P_h$  and  $(1 - P_h)$  of (i) and (ii) respectively,

The probability of a "Yes" answer for  $j^{th}$  characteristics;  $j = 1, 2, \dots, p$  in the  $h^{th}$  stratum is given by

$$Y_{jh} = M_h \pi_{shj} + (1 - M_h) [P_h \pi_{shj} + (1 - P_h)(1 - \pi_{shj})]; \quad (6.1)$$

$$h = 1, 2, \dots, L; j = 1, 2, \dots, p.$$

and  $\pi_{shj}$  is the proportion of respondents belonging to the sensitive group for  $j^{th}$  characteristics in the  $h^{th}$  stratum. The maximum likelihood estimate of  $\pi_{shj}$  is given as

$$\hat{\pi}_{shj} = \frac{\hat{Y}_{hj} - (1 - M_h)(1 - P_h)}{2P_h - 1 + 2M_h(1 - P_h)}; \quad h = 1, 2, \dots, L; j = 1, 2, \dots, p. \quad (6.2)$$

where  $\hat{Y}_{hj}$  is the estimated proportion of "Yes" answers which follows a binomial distribution

$B(n_h, Y_{hj})$  and  $n_h$  denote the sample size from  $h^{th}$  stratum.

It can be seen that the estimator  $\hat{\pi}_{shj}$  is unbiased with variance  $V(\hat{\pi}_{shj})$

Expressions (6.1) and (6.2) are from Mangat and Singh (1990) with added suffix ' $h$ ' to denote the  $h^{th}$  stratum;  $j = 1, 2, \dots, L$  and  $j$  to denote the  $j^{th}$  characteristic;  $j = 1, 2, \dots, p$ . Since  $n_h$  are drawn independently from each stratum, the estimators for individual strata can be added to obtain the estimator for the overall population parameter. This gives an unbiased estimate of  $\pi_j$  by using Eqn. (6.2) as:



$$\hat{\pi}_{sj} = \sum_{h=1}^L W_h \hat{\pi}_{shj} = \sum_{h=1}^L W_h \left[ \frac{\hat{Y}_{hj} - (1 - M_h)(1 - P_h)}{2P_h - 1 + 2M_h(1 - P_h)} \right], \quad (6.3)$$

where  $W_h; h=1, 2, \dots, L$  are the strata weights.

The sampling variance of  $\hat{\pi}_{sj}$  (see Mangat and Singh (1990)) define as

$$\left. \begin{aligned} V(\hat{\pi}_{sj}) &= \sum_{h=1}^L W_h^2 V(\hat{\pi}_{shj}), \quad j=1, 2, \dots, p. \\ &= \sum_{h=1}^L \frac{W_h^2}{n_h} \left\{ \pi_{shj}(1 - \pi_{shj}) + \frac{(1 - M_h)(1 - P_h)[1 - (1 - M_h)(1 - P_h)]}{[2P_h - 1 + 2M_h(1 - P_h)]^2} \right\} \\ &\Rightarrow \sum_{h=1}^L \frac{a_{hj}}{n_h}, \quad j=1, 2, \dots, p, \\ \text{where } a_h &= W_h^2 \left\{ \pi_{shj}(1 - \pi_{shj}) + \frac{(1 - M_h)(1 - P_h)[1 - (1 - M_h)(1 - P_h)]}{[2P_h - 1 + 2M_h(1 - P_h)]^2} \right\} \end{aligned} \right\} \quad (6.4)$$

### 6.3 Geometric Programming Formulation

The following multi-objective integer nonlinear programming problem (MINLPP) with linear cost function and the cost function quadratic in  $\sqrt{n_h}$  and significant travel cost are define in equations (6.5) and (6.8) respectively (See Ghufraan *et al.* (2012, 2013) for more details) define as:

**For linear cost function**

$$\left. \begin{aligned} \min V(\hat{\pi}_{sj}) &= \sum_{h=1}^L \frac{a_{hj}}{n_h} \\ \text{subject to } \sum_{h=1}^L c_h n_h &\leq C - c_0 = C_0, \quad j=1, 2, \dots, p \\ n_h &\geq 0 \text{ and } h=1, 2, \dots, L; \end{aligned} \right\} \quad (6.5)$$

The above expression (6.5) can be expressed in the standard primal GPP with the linear cost function is given as follows:

Mangat (1996). Greenberg *et al.* (1969), Kuk (1990), Mangat *et al.* (1995), Mahmood *et al.* (1998), Bhargava and Singh (2000), Singh (2002,2003), Singh *et al.* (2001), Chang and Huang (2001), Gupta *et al.* (2002), Christofides (2003), Kim and Warde (2004), Huang (2006), Hussain *et al.* (2007), Hussain and Shabbir (2010). Chaudhuri and Saha (2005), Franklin (1989), Diana and Perri (2009), Horvitz *et al.* (1967), Moors (1971), Ghufraan *et al.* (2012, 2013) have formulated the Two-Stage Stratified Randomized Response Model as a mathematical programming problem and obtained the compromise allocations.

Geometric Programming (GP) has been known as an optimization tools for solving the problems in various fields. Duffin *et al.* (1967) and also Zener (1971) have discussed the basic concepts and theories of GP with application in engineering in their books. Davis and Rudolph (1987) applied GP to optimal allocation of integrated samples in quality control. Ahmed and Charles (1987) applied geometric programming to obtain the optimum allocation problems in multivariate double sampling. Maqbool *et al.* (2011), Shafiullah *et al.* (2013) have discussed the geometric programming approach to find the optimum allocations in multivariate two-stage sampling and three-stage sample surveys respectively.

The concept of fuzzy set theory was first of all given by Zadeh (1965). Later on, Bellman and Zadeh (1970) used the fuzzy set theory to the decision-making problem. Tanaka (1974) introduced the objective as fuzzy goal over the  $\alpha$ -cut of a fuzzy constraint set and Zimmermann (1978) gave the concept to solve multi-objective linear-programming problem. Biswal (1992) and Verma (1990) developed fuzzy geometric programming technique to solve multi-objective geometric programming (MOGP) problem. Islam (2005, 2010) has discussed modified geometric programming problem and its applications and also another fuzzy geometric programming technique to solve MOGPP and their applications. Fuzzy mathematical programming has been applied to several fields.

In this chapter the problem of obtaining the optimum allocation for the two-stage stratified Warner's *RR* model has been discussed for multiple characteristics in two different situations, the linear cost function and the cost function quadratic in  $\sqrt{n_h}$  with two variables and four variables. The fuzzy geometric programming procedure has been described for solving the formulated MOGPP for the linear and quadratic cost function.

$$\left. \begin{array}{l} \min f_{0j}(n), \\ \text{subject } f_q(n) \leq 1 \\ n_h \geq 0, \quad h=1,2,\dots,L \end{array} \right\}, j=1,2,\dots,p \quad (6.6)$$

where  $f_{0j}(n) = V(\hat{\pi}_{sj}) = \sum_{h=1}^L \frac{a_{hj}}{n_h}$ ,  $j=1,2,\dots,p$ .

$$f_q(n) = \sum_{i \in j[q]} d_i n_1^{p_{i1}} n_2^{p_{i2}} \dots n_L^{p_{iL}}, q=0,1,2,\dots,k$$

$$\text{or } f_q(n) = \sum_{i \in j[q]} d_i \left[ \prod_{h=1}^L n_h^{p_{ih}} \right], d_i > 0, n_h > 0, q=0,1,\dots,k$$

$p_{ih}$ : arbitrary real numbers,  $d_i$ : positive and  $f_q(n)$ : posinomials

$d_i = a_{ij}$  for objective function and  $d_i = \frac{c_h}{C_0}$  for constraints.

Let  $m$  denotes the number of posynomial terms in the function,  $h$  is the number of variables and the exponent  $P_{ih}$  are real constants.

The dual form of the primal GPP which is stated in (6.6) can be given as:

$$\left. \begin{array}{l} \max v_{0j}(w) = \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject to } \sum_{i \in [0]} w_i = 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i = 0 \quad (iii) \\ w_i \geq 0, q=0,1,\dots,k \text{ and } i=1,2,\dots,m_k \quad (iv) \end{array} \right\}, j=1,\dots,p. \quad (6.7)$$

The cost function quadratic in  $\sqrt{n_h}$

$$\left. \begin{array}{l} \min V(\hat{\pi}_{sj}) = \sum_{h=1}^L \frac{a_{hj}}{n_h} \\ \text{subject to} \\ \sum_{h=1}^L c_h n_h + \sum_{h=1}^L t_h \sqrt{n_h} \leq C - c_0 = C_0, \\ n_h \geq 0 \text{ and } h=1,2,\dots,L. \end{array} \right\}, j=1,2,\dots,p \quad (6.8)$$

Similarly, the expression (6.8) can be expressed in the standard primal GPP with cost function quadratic in  $\sqrt{n_h}$  where the travel cost is significant can be written as follows:

$$\left. \begin{array}{l} \max f_{0j}(n) \\ \text{subject } f_q(n) \leq 1 \\ n_h \geq 0, \quad h=1,2,\dots,L \end{array} \right\}, j=1,2,\dots,p \quad (6.9)$$

where  $f_q(n) = \sum_{i \in j[q]} d_i n_1^{p_{i1}} n_2^{p_{i2}} \dots n_L^{p_{iL}}, q=0,1,2,\dots,k$

or  $f_q(n) = \sum_{i \in j[q]} d_i \left[ \prod_{h=1}^L n_h^{p_{ih}} \right], d_i > 0, n_h > 0, q=0,1,2,\dots,k,$

$p_{ih}$ : arbitrary real numbers,  $d_i$ : positive and  $f_q(n)$ : posinomials

Let for simplicity  $d_i = a_{ij} = \frac{c_h}{C_0} = \frac{t_h}{C_0}$

The dual form of the primal GPP which is stated in (6.9) can be given as:

$$\left. \begin{array}{l} \max v_{0j}(w) = \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject to } \sum_{i \in [0]} w_i = 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i = 0 \quad (iii) \\ w_i \geq 0, q=0,1,\dots,k \text{ and } i=1,2,\dots,m_k \quad (iv) \end{array} \right\}, j=1,\dots,p. \quad (6.10)$$

The above formulated GPP (6.7) and (6.10) can be solved in the following two-steps:

**Step 1:** For the Optimum value of the objective function, the objective function always takes the form:

$$\begin{aligned} C_0(x^*) &= \left( \frac{\text{Coeffi. of first term}}{w_{01}} \right)^{w_{01}} \times \left( \frac{\text{Coeffi. of second term}}{w_{02}} \right)^{w_{02}} \\ &\times \dots \times \left( \frac{\text{Coeffi. of last term}}{w_k} \right)^{w_{0k}} \left( \sum w' \text{ s in the first constraints} \right)^{\sum w' \text{ s in the first constraints}} \\ &\left( \sum w' \text{ s in the last constraints} \right)^{\sum w' \text{ s in the last constraints}} \end{aligned}$$

The Multi-Objective objective function for our problem is:

$$\prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i}$$

**Step 2:** The equations that can be used for GPP for the weights are given below:

$$\sum_{i \in J[0]} w_i \text{ in the objective function} = 1 \text{ (Normality condition)}$$

and for each primal variable  $n_h$  and  $\sqrt{n_h}$  having  $m$  terms.

$$\sum_{i=1}^{m_k} (w_i \text{ for each term}) \times \left( \text{exponent on } n_h \& \sqrt{n_h} \text{ in that term} \right) = 0 \text{ (Orthogonality condition)}$$

and  $w_i \geq 0$  (Positivity condition).

The above problems (6.7) and (6.10) have been solved with the help of steps (1-2) discussed in section (6.3) and the corresponding solution  $w_{0i}^*$  is the unique solution to the dual constraints; it will also maximize the objective function for the dual problem. Next, the solution of the primal problem will be obtained using primal-dual relationship theorem which is given below:

**6.4 Primal-dual relationship theorem:** If  $w_{0i}^*$  is a maximizing point for dual problem (6.7) and (6.10), each optimal values of the two-stage randomized response model which is the minimizing points  $(n^*)$  for the primal GPP's (6.6) and (6.9) satisfies the system of equations:

$$f_{0j}(n) = \begin{cases} w_{0i}^* v(w^*), & i \in J[0] \\ \frac{w_{ij}}{v_L(w_{0i}^*)}, & i \in J[L] \end{cases} \quad (6.11)$$

where  $L$  ranges over all positive integers for which  $v_L(w_{0i}^*) > 0$ .

The optimal values of the sample sizes of the problems  $n_h^*$  can be calculated with the help of the primal-dual relationship theorem (6.11).

## 6.5 Fuzzy Geometric Programming Approach

The solution procedure to solve the problem (6.15) consists of the following steps:

Step-1: Solve the MOGPP as a single objective problem using only one objective at a time and ignoring the others. These solutions are known as ideal solution.

Step-2: From the results of step-1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, pay-off matrix can be formulated as follows:

$$\begin{array}{cccccc}
& f_{01}(n) & f_{02}(n) & \cdots & f_{0j}(n) & \cdots & f_{0p}(n) \\
(n^{(1)}) & \left[ f_{01}^*(n^{(1)}) & f_{02}(n^{(1)}) & \cdots & f_{0j}(n^{(1)}) & \cdots & f_{0p}(n^{(1)}) \right. \\
(n^{(2)}) & f_{01}(n^{(2)}) & f_{02}^*(n^{(2)}) & \cdots & f_{0j}(n^{(2)}) & \cdots & f_{0p}(n^{(2)}) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
(n^{(j)}) & f_{01}(n^{(j)}) & f_{02}(n^{(j)}) & \cdots & f_{0j}^*(n^{(j)}) & \cdots & f_{0p}(n^{(j)}) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
(n^{(p)}) & f_{01}(n^{(p)}) & f_{02}(n^{(p)}) & \cdots & f_{0j}(n^{(p)}) & \cdots & f_{0p}^*(n^{(p)})
\end{array}$$

Here  $(n^{(1)}), (n^{(2)}), \dots, (n^{(j)}), \dots, (n^{(p)})$  are the ideal solutions of the objective functions  $f_{01}(n^{(1)}), f_{02}(n^{(2)}), \dots, f_{0j}(n^{(j)}), \dots, f_{0p}(n^{(p)})$ .

So  $U_j = \text{Max} \{f_{01}(n^{(1)}), f_{02}(n^{(2)}), \dots, f_{0p}(n^{(p)})\}$  and  $L_j = f_{0j}^*(n^{(j)}), j=1,2,\dots, p$ .

$$\left[ \begin{array}{l} U_j \text{ and } L_j \text{ be the upper and lower bonds of the } j^{\text{th}} \text{ objective function} \\ f_{0j}(n), j=1,2,\dots, p. \end{array} \right]$$

Step 3: The membership function for the given problem can be defined as:

$$\mu_j(f_{0j}(n)) = \begin{cases} 0, & \text{if } f_{0j}(n) \geq U_j \\ \frac{U_j(n) - f_{0j}(n)}{U_j(n) - L_j(n)}, & \text{if } L_j \leq f_{0j}(n) \leq U_j, j=1,2,\dots, p \\ 1, & \text{if } f_{0j}(n) \leq L_j \end{cases} \quad (6.12)$$

Here  $U_j(n)$  is a strictly monotonic decreasing function with respect to  $f_{0j}(n)$ .

Following figure illustrates the graph of the membership function  $\mu_j(f_{0j}(n))$ .

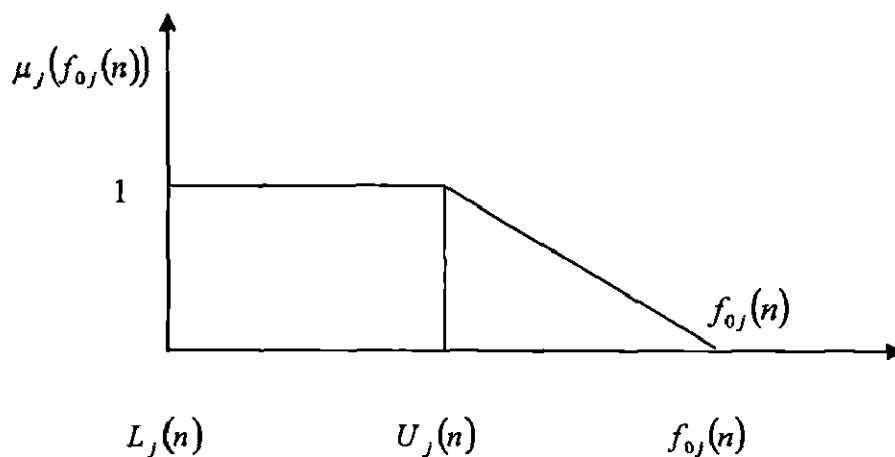


Figure-3.1 Membership function for minimization variances problem

The membership functions in Eqn. (6.12)

i.e.,  $\mu_j(f_{0j}(n)), j=1, 2, \dots, p$

Therefore the general aggregation function can be defined as

$$\mu_{\bar{D}}(n) = \mu_{\bar{D}} \{ \mu_1(f_{01}(n)), \mu_2(f_{02}(n)), \dots, \mu_p(f_{0p}(n)) \}$$

The fuzzy multi-objective formulation of the problem with linear cost function can be defined as:

$$\left. \begin{array}{l} \max \mu_{\bar{D}}(n) \\ \text{subject to } \sum_{h=1}^L c_h n_h \leq C - c_0 = C_0; \\ n_h \geq 0 \text{ and } h=1, 2, \dots, L. \end{array} \right\} \quad (6.13)$$

The problem to find the optimal values of  $(n^*)$  for this convex-fuzzy decision based on addition operator (like Tiwari *et. al.* (1987)). Therefore the problem (6.13) is reduced according to max-addition operator as

$$\left. \begin{array}{l} \max \mu_{\bar{D}}(n^*) = \sum_{j=1}^p \mu_j(f_{0j}(n)) = \sum_{j=1}^p \frac{U_j - (f_{0j}(n))}{U_j - L_j} \\ \text{subject to } \sum_{h=1}^L c_h n_h \leq C_0; \\ 0 \leq \mu_j(f_{0j}(n)) \leq 1, \\ n_h \geq 0 \text{ and } h=1, 2, \dots, L. \end{array} \right\} \quad (6.14)$$

The problem (6.14) reduces to

$$\left. \begin{array}{l} \max \mu_{\bar{D}}(n^*) = \sum_{j=1}^p \left\{ \frac{U_j}{U_j - L_j} - \frac{(f_{0j}(n))}{U_j - L_j} \right\} \\ \text{subject to } f_q(n) \leq 1; \\ n_h \geq 0 \text{ and } j=1, 2, \dots, p. \end{array} \right\} \quad (6.15)$$

$$\text{where } f_q(n) = \sum_{h=1}^L \frac{c_h}{C_0} n_h$$

The problem (6.15) maximizes if the function  $F_{0j}(n) = \left\{ \frac{(f_{0j}(n))}{U_j - L_j} \right\}$  attain the minimum values. Therefore the problem (6.15) reduces into the primal problem (6.16) defined as:

$$\left. \begin{array}{l} \min \sum_{j=0}^p F_{oj}(n) \\ \text{subject to} \\ f_q(n) \leq 1; \\ n_h \geq 0 \text{ and } j=1,2,\dots,p \end{array} \right\} \quad (6.16)$$

The dual form of the primal GPP which is stated in (6.16) can be given as:

$$\left. \begin{array}{l} \max v(w) = \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject to } \sum_{i \in [0]} w_i = 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i = 0 \quad (iii) \\ w_i \geq 0, q=0,1,\dots,k \text{ and } i=1,2,\dots,m_k \quad (iv) \end{array} \right\} \quad (6.17)$$

Similarly, the fuzzy multi-objective formulation of the problem with cost function quadratic in  $\sqrt{n_h}$  and significant travel cost can be defined as:

$$\left. \begin{array}{l} \min \sum_{j=0}^p F_{oj}(n') \\ \text{subject to} \\ f_q(n') \leq 1; \\ n_h \geq 0 \text{ and } j=1,2,\dots,p. \end{array} \right\} \quad (6.18)$$

$$\text{where } f_q(n) = \sum_{h=1}^L \frac{c_h}{C_0} n'_h + \sum_{h=1}^L \frac{t_h}{C_0} \sqrt{n'_h}$$

The dual form of the primal GPP which is stated in (6.18) can be given as:

$$\left. \begin{array}{l} \max v(w) = \prod_{q=0}^k \prod_{i \in j[q]} \left\{ \left( \frac{d_i}{w_i} \right)^{w_i} \right\} \prod_{q=1}^k \left( \sum_{i \in j[q]} w_i \right)^{\sum_{i \in j[q]} w_i} \quad (i) \\ \text{subject } \sum_{i \in [0]} w_i = 1 \quad (ii) \\ \sum_{q=0}^k \sum_{i \in j[q]} p_{ih} w_i = 0 \quad (iii) \\ w_i \geq 0, q=0,1,\dots,k \text{ and } i=1,2,\dots,m_k \quad (iv) \end{array} \right\} \quad (6.19)$$

The optimal values of the sample sizes of the problems  $n_h^*$  can be calculated with the help of the primal–dual relationship theorem (6.11).



## 6.6 Numerical illustrations:

**Example 1:** The following data are from Ghufuran *et al.* (2012). The population size  $N$  is assumed to be 1,000. This gives  $N_1 = 350$  and  $N_2 = 650$ . It is also assumed that the total budget of the survey  $C=4,500$  units with an overhead cost  $c_0 = 500$  units. Thus  $C_0 = C - c_0 = 4,500 - 500 = 4,000$  units are available for measurements. Using the values given in Table 1 is as  $A_1 = 0.060945273$  and  $A_2 = 0.072830578$ .

**Table 1: Data for two characteristics and two Strata**

$h$	$W_h$	$\pi_{sh1}$	$\pi_{sh2}$	$M_h$	$P_h$	Travel cost is not significant $c_h$	Travel cost is significant $c'_h$ $t'_h$
(1)	(2)	(3)	(4)	(7)	(8)		
1	0.35	0.45	0.35	0.90	0.80	25	15 10
2	0.65	0.85	0.75	0.80	0.70	33	20 13

For solving this MOGP problem, we shall first solve the two sub-problems:

**Sub problem1:** On substituting the table values in sub problem 1, we have obtained the expressions given below:

$$\left. \begin{array}{l} \min f_{01} = \frac{0.037784545}{n_1} + \frac{0.084639669}{n_2} \\ \text{subject to} \\ 0.00625 n_1 + 0.00825 n_2 \leq 1; \\ n_1 \geq 0, n_2 \geq 0; \end{array} \right\} \quad (6.20)$$

The dual of the above problem (6.20) is given below:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left( (0.037784545/w_{01})^{w_{01}} \right) \times \left( (0.084639669/w_{02})^{w_{02}} \right) \times \\
 & \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \\
 & ((w_{11} + w_{12})^{(w_{11} + w_{12})}); \quad (i) \\
 \text{subject to} \quad & w_{01} + w_{02} = 1; \quad (\text{normality condition}) \quad (ii) \\
 & \left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{02} + w_{12} &= 0 \end{aligned} \right\} \quad (\text{orthogonality condition}) \quad (iii) \\
 & w_{01}, w_{02}, w_{11}, w_{12} \geq 0 \quad (\text{positivity condition}) \quad (iv)
 \end{aligned} \tag{6.21}$$

For orthogonality condition defined in expression 6.21(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$\begin{aligned}
 n^{(i)} \Rightarrow w_{01} &= 0.3677068, w_{02} = 0.6322932, w_{11} = 0.3677068, w_{12} = 0.6322932 \\
 \text{and } w(V_1) &= 0.001746590
 \end{aligned}$$

The optimal values  $n_h^*$  of the sample sizes of the standard primal problems can be calculated with the help of the primal–dual relationship theorem (6.11).

$$f_{0j}(n) = w_{0i}^* v(w^*)$$

$$f_{01}(n_1) = w_{01}^* v(w^*) \Rightarrow n_1 \cong 59 \quad f_{02}(n_2) = w_{02}^* v(w^*) \Rightarrow n_2 \cong 77$$

The optimal values and the objective function value are given below:

$$n_1^* = 59, n_2^* = 77 \quad \text{and the objective value of the primal problem is } 0.001746590$$

**Sub problem 2:** On substituting the table values in sub problem 2, we have obtained the expressions given below:

$$\begin{aligned}
 \min \quad & f_{02} = \frac{0.035334545}{n_1} + \frac{0.109989669}{n_2} \\
 \text{subject to} \quad & 0.00625 n_1 + 0.00825 n_2 \leq 1; \\
 & n_1 \geq 0, n_2 \geq 0;
 \end{aligned} \tag{6.22}$$

The dual of the above formulated problem (6.22) is given in Eqn. (6.23) as follows:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left( (0.035334545/w_{01})^{w_{01}} \right) \times \left( (0.109989669/w_{02})^{w_{02}} \right) \times \\
 & \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \\
 & ((w_{11} + w_{12})^{w_{11} + w_{12}}); \quad (i) \\
 \text{subject to} \quad & w_{01} + w_{02} = 1; \quad (\text{normality condition}) \quad (ii) \\
 & \left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{11} + w_{12} &= 0 \end{aligned} \right\} \quad (\text{orthogonality condition}) \quad (iii) \\
 & w_{01}, w_{02}, w_{11}, w_{12} \geq 0 \quad (\text{positivity condition}) \quad (iv)
 \end{aligned} \tag{6.23}$$

For orthogonality condition defined in expression 6.23 (iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$\begin{aligned}
 n^{(1)} \Rightarrow w_{01} &= 0.3303553, w_{02} = 0.6696447, w_{11} = 0.3303553, w_{12} = 0.6696447 \\
 \text{and } w(V_2) &= 0.002023564
 \end{aligned}$$

The optimal values  $n_h^*$  of the sample sizes of the standard primal problems can be calculated with the help of the primal–dual relationship theorem (6.11).

$$f_{0j}(n) = w_{0i}^* v(w^*)$$

$$f_{01}(n_1) = w_{01}^* v(w^*) \Rightarrow n_1 \cong 53 \quad f_{02}(n_2) = w_{02}^* v(w^*) \Rightarrow n_2 \cong 81$$

The optimal values and the objective function value are given below:

$$n_1^* = 53, n_2^* = 81 \text{ and the objective value of the primal problem is } 0.002023564.$$

	$f_{01}(n)$	$f_{02}(n)$
$n^{(1)}$	0.001746590	0.002035707
$n^{(2)}$	0.001757605	0.002023564

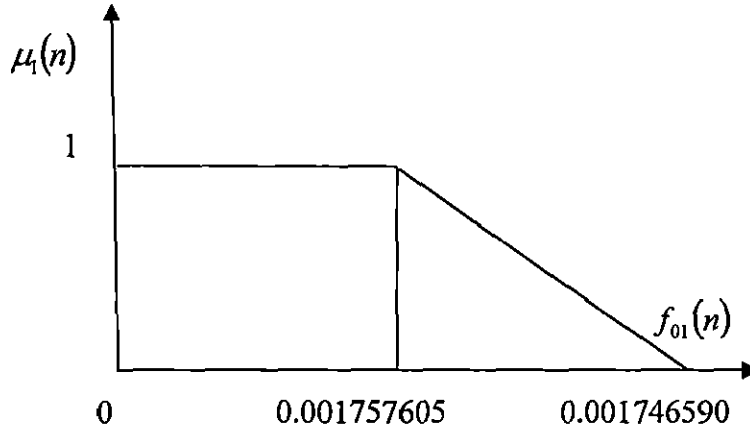
The lower and upper bond of  $f_{01}(n)$  and  $f_{02}(n)$  can be obtained from the pay-off matrix

$$0.001746590 \leq f_{01}(n) \leq 0.001757605 \text{ and } 0.002023564 \leq f_{02}(n) \leq 0.002035707.$$

Let  $\mu_1(n)$  and  $\mu_2(n)$  be the fuzzy membership function of the objective function  $f_{01}(n)$  and  $f_{02}(n)$  respectively and they are defined as:

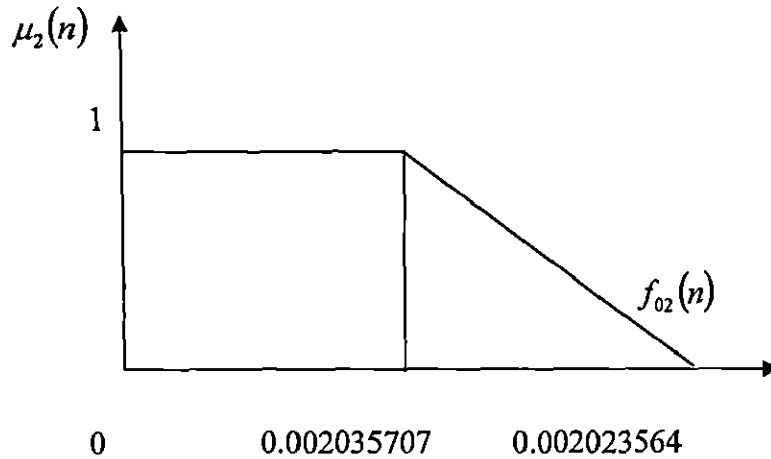
$$\mu_1(n) = \begin{cases} 1 & , \text{if } f_{01}(n) \leq 0.001746590 \\ \frac{0.001757605 - f_{01}(n)}{0.000011015} & , \text{if } 0.001746590 \leq f_{01}(n) \leq 0.001757605 \\ 0 & , \text{if } f_{01}(n) \geq 0.001757605 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_2(n) = \begin{cases} 1 & , \text{if } f_{02}(n) \leq 0.002023564. \\ \frac{0.002035707 - f_{02}(n)}{0.000012143} & , \text{if } 0.002023564 \leq f_{02}(n) \leq 0.002035707 \\ 0 & , \text{if } f_{02}(n) \geq 0.002035707. \end{cases}$$

Now the following figure illustrated the graph of the fuzzy membership function  $\mu_2(n)$



On applying the max-addition operator, the MOGPP, reduces to the crisp problem as:

$$\begin{aligned}
& \max (\mu_1(n) + \mu_2(n)) \\
& \text{e max } \left\{ \frac{0.00175605 - f_{01}(n)}{0.000011015} + \frac{0.002035707 - f_{02}(n)}{0.000012143} \right\} \\
& \text{e max } \left\{ 327.06799 - \left( \frac{f_{01}(n)}{0.000011015} + \frac{f_{02}(n)}{0.000012143} \right) \right\} \\
& \text{subject to} \\
& \quad 0.00625 n_1 + 0.00825 n_2 \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned} \tag{6.24}$$

In order to maximize the problem (6.24), we have to minimize

$\left( \frac{f_{01}(n)}{0.000011015} + \frac{f_{02}(n)}{0.000012143} \right)$  as follows:

$$\begin{aligned}
& \min \left( \frac{f_{01}(n)}{0.000011015} + \frac{f_{02}(n)}{0.000012143} \right) \\
& \text{or min } (14801.65779 \times f_{01}(n) + 79634.3192 \times f_{02}(n)) \\
& \text{or min } \left\{ \left( \frac{90785.29278 \times 0.037784545}{n_1} + \frac{90785.29278 \times 0.084639669}{n_2} \right) + \right. \\
& \quad \left. \left( \frac{82351.97233 \times 0.035334545}{n_1} + \frac{82351.97233 \times 0.109989669}{n_2} \right) \right\} \\
& \text{e. min } \left\{ \frac{6340.15045}{n_1} + \frac{16741.90331}{n_2} \right\} \\
& \text{subject to} \\
& \quad 0.00625 n_1 + 0.00825 n_2 \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned}$$

So, our new problem for the solution is given as follows:

$$\begin{aligned}
& \min \left\{ \frac{6340.15045}{n_1} + \frac{16741.90331}{n_2} \right\} \\
& \text{subject to} \\
& \quad 0.00625 n_1 + 0.00825 n_2 \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned} \tag{6.25}$$

Degree of Difficulty of the problem (6.25) is  $= (4 - (2 + 1)) = 1$

Hence the dual problem of the above problem (6.25) is given as:

$$\begin{aligned}
& \max v(w_{0i}^*) = \left[ \left( (6340.15045/w_{01})^{w_{01}} \right) \times \left( (16741.90328/w_{02})^{w_{02}} \right) \times \right. \\
& \left. \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \right. \\
& \left. ((w_{11} + w_{12})^{(w_{11} + w_{12})}); \right. \quad (i) \\
& \text{subject to } w_{01} + w_{02} = 1; \text{ (normality condition)} \quad (ii) \\
& \quad \left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{11} + w_{12} &= 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
& \quad w_{01}, w_{02}, w_{11}, w_{12} \geq 0 \quad \text{(positivity condition)} \quad (iv)
\end{aligned} \tag{6.26}$$

For orthogonality condition defined in expression 6.26 (iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \end{cases}$$

After solving the formulated dual problem (6.26) using lingo software we obtain the following values of the dual variables which are given as:

$$w_{01} = 0.3487991, w_{02} = 0.6512009 \text{ and } v(w_{0i}^*) = 325.7083$$

The optimal values  $n_h^*$  of the sample sizes of the standard primal problems can be calculated with the help of the primal–dual relationship theorem (6.11).

$$f_{0j}(n) = w_{0i}^* v(w^*)$$

$$f_{01}(n_1) = w_{01}^* v(w^*) \Rightarrow n_1 \cong 56$$

$$f_{02}(n_2) = w_{02}^* v(w^*) \Rightarrow n_2 \cong 79$$

The optimal values and the objective function value are given below:

$$n_1^* = 56, n_2^* = 79 \text{ and the objective value of the primal problem is } 325.7083.$$

### The MOGP problem with quadratic cost function:

**Sub problem1:** On substituting the values from table 1 in sub problem 1, we have obtained the expressions given below:

$$\begin{aligned}
& \min f_{01} = \frac{0.037784545}{n_1} + \frac{0.084639669}{n_2} \\
& \text{subject to} \\
& \quad 0.00375n_1 + 0.005n_2 + 0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned} \tag{6.27}$$

The dual of the above formulated problem (6.27) is given in Eqn. (6.28) as follows:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left( (0.037784545/w_{01})^{w_{01}} \right) \times \left( (0.084639669/w_{02})^{w_{02}} \right) \times \\
 & \left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0025}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.00325}{w_{14}} \right)^{w_{14}} \right) \\
 & ((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
 \text{subject to} \quad & w_{01} + w_{02} = 1; \text{ (normality condition)} \quad (ii) \\
 & \left. \begin{aligned} -w_{01} + w_{11} + (1/2)w_{13} &= 0 \\ -w_{11} + w_{12} + (1/2)w_{14} &= 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
 & w_{01}, w_{02}, w_{11}, w_{12}, w_{13}, w_{14} \geq 0 \quad \text{ (positivity condition)} \quad (iv)
 \end{aligned} \quad (6.28)$$

For orthogonality condition defined in expression 6.28(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{13} = 0 \\ -w_{02} + w_{12} + (1/2)w_{14} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$n^{(2)} \Rightarrow w_{01} = 0.3671054, w_{02} = 0.6328946, w_{11} = 0.3547625, w_{12} = 0.6146045, w_{13} = 0.0246857, w_{14} = 0.03658035 \text{ and } w(V_2) = 0.001121298$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem1 are given as follows:

$$n_1^* = 92, n_2^* = 119 \text{ and the objective value of the primal problem is } 0.001121298.$$

**Sub problem 2:** On substituting the values from table 1 in sub problem 2, we have obtained the expressions given below:

$$\begin{aligned}
 \min \quad & f_{02} = \frac{0.035334545}{n_1} + \frac{0.109989669}{n_2} \\
 \text{subject to} \quad & 0.00375 n_1 + 0.005 n_2 + 0.0025 \sqrt{n_1} + 0.00325 \sqrt{n_2} \leq 1 \\
 & n_1 \geq 0, n_2 \geq 0;
 \end{aligned} \quad (6.29)$$

The dual of the above formulated problem (6.29) is given in Eqn. (6.30) as follows:

$$\begin{aligned}
& \max v(w_{0i}^*) = \left\{ \left( (0.035334545/w_{01})^{w_{01}} \right) \times \left( (0.109989669/w_{02})^{w_{02}} \right) \times \right. \\
& \left. \left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0025}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.00325}{w_{14}} \right)^{w_{14}} \right) \right\} \\
& ((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
& \text{subject to } w_{01} + w_{02} = 1; \quad (\text{normality condition}) \quad (ii) \\
& \left. \begin{aligned} & -w_{01} + w_{11} + (1/2)w_{13} = 0 \\ & -w_{02} + w_{12} + (1/2)w_{14} = 0 \end{aligned} \right\} \quad (\text{orthogonality condition}) \quad (iii) \\
& w_{01}, w_{02}, w_{11}, w_{12}, w_{13}, w_{14} \geq 0 \quad (\text{positivity condition}) \quad (iv)
\end{aligned} \quad (6.30)$$

For orthogonality condition defined in expression 6.30(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{13} = 0; \\ -w_{02} + w_{12} + (1/2)w_{14} = 0; \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$\begin{aligned}
n^{(2)} \Rightarrow w_{01} &= 0.3300730, w_{02} = 0.6699270, w_{11} = 0.3183804, w_{12} = 0.6511027, \\
w_{13} &= 0.02338512, w_{14} = 0.03764872, \text{ and } w(V_2) = 0.00129930
\end{aligned}$$

The optimal values  $n_i^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 82, n_2^* = 126 \text{ and the objective value of the primal problem is } 0.00129930.$$

Now the pay-off matrix of the above problems is given below:

$$\begin{matrix} & f_{01}(n^{(1)}) & f_{02}(n^{(2)}) \\ \begin{matrix} n^{(1)} \\ n^{(2)} \end{matrix} & \begin{bmatrix} 0.001121298 & 0.001307155 \\ 0.001128421 & 0.00129930 \end{bmatrix} \end{matrix}$$

The lower and upper bond of  $f_{01}(n)$  and  $f_{02}(n)$  can be obtained from the pay-off matrix

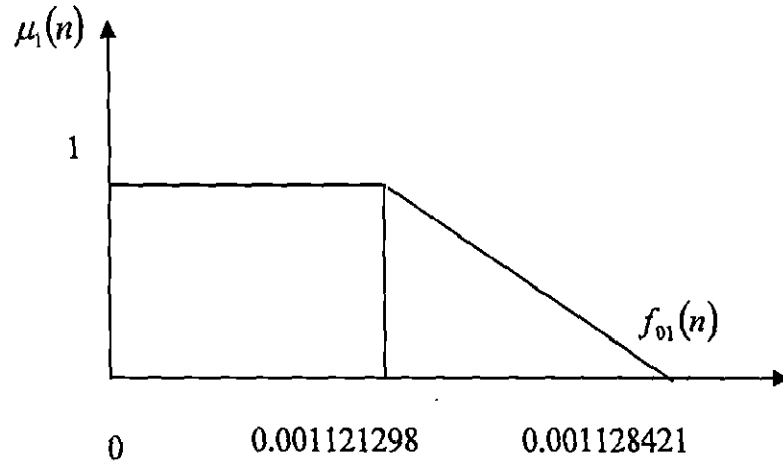
$$0.001121298 \leq f_{01}(n) \leq 0.001128421 \text{ and } 0.00129930 \leq f_{02}(n) \leq 0.001307155.$$

Let  $\mu_1(n)$  and  $\mu_2(n)$  be the fuzzy membership function of the objective function  $f_{01}(n)$  and  $f_{02}(n)$  respectively and they are defined as:



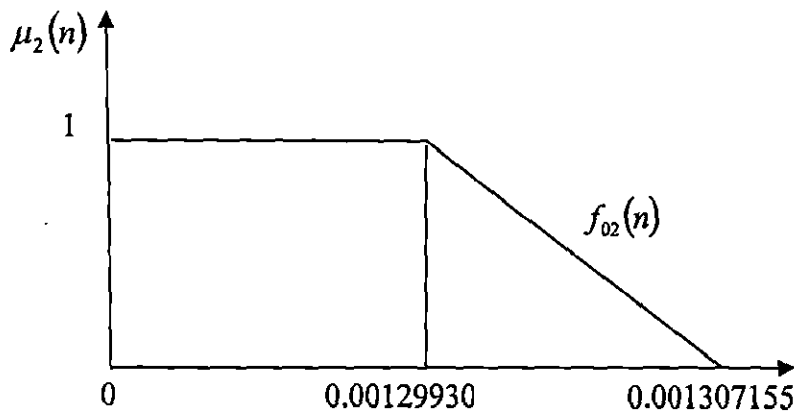
$$\mu_1(n) = \begin{cases} 1 & , \text{if } f_{01}(n) \leq 0.001121298 \\ \frac{0.001128421 - f_{01}(n)}{0.000007123} & , \text{if } 0.001121298 \leq f_{01}(n) \leq 0.001128421 \\ 0 & , \text{if } f_{01}(n) \geq 0.001128421 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_2(n) = \begin{cases} 1 & , \text{if } f_{02}(n) \leq 0.00129930 \\ \frac{0.001307155 - f_{02}(n)}{0.000007855} & , \text{if } 0.00129930 \leq f_{02}(n) \leq 0.001307155 \\ 0 & , \text{if } f_{02}(n) \geq 0.001307155 \end{cases}$$

Now the following figure illustrated the graph of the fuzzy membership function  $\mu_2(n)$



On applying the max-addition operator, the final primal MOGPP with cost function quadratic in  $\sqrt{n_h}$  and significant travel cost can be expressed according to the described procedure in sub- problem1 is given as follows:

$$\begin{aligned}
& \max \quad (\mu_1(n) + \mu_2(n)) \\
& i.e \max \left\{ \frac{0.001128421 - f_{01}(n)}{0.000007123} + \frac{0.001307155 - f_{02}(n)}{0.000007855} \right\} \\
& i.e \max \left\{ 324.8299 - \left( \frac{f_{01}(n)}{0.000007123} + \frac{f_{02}(n)}{0.000007855} \right) \right\} \\
& \text{subject to} \\
& \quad 0.00625n_1 + 0.00825n_2 \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned} \tag{6.31}$$

In order to maximize the problem (6.31), we have to minimize

$$\left( \frac{f_{01}(n)}{0.000007123} + \frac{f_{02}(n)}{0.000007855} \right) \text{ as follows:}$$

$$\begin{aligned}
& \min \left( \frac{f_{01}(n)}{0.000007123} + \frac{f_{02}(n)}{0.000007855} \right) \\
& \text{or } \min (140390.285 \times f_{01}(n) + 127307.4475 \times f_{02}(n)) \\
& \text{or } \min \left\{ \left( \frac{140390.285 \times 0.037784545}{n_1} + \frac{140390.285 \times 0.084639669}{n_2} \right) + \right. \\
& \quad \left. \left( \frac{127307.4475 \times 0.035334545}{n_1} + \frac{127307.4475 \times 0.109989669}{n_2} \right) \right\} \\
& i.e. \min \left\{ \frac{9802.9337}{n_1} + \frac{25885.09126}{n_2} \right\} \\
& \text{subject to} \\
& \quad 0.00375n_1 + 0.005n_2 + 0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned}$$

$$\begin{aligned}
& \min \left\{ \frac{9802.9337}{n_1} + \frac{25885.0913}{n_2} \right\} \\
& \text{subject to} \\
& \quad 0.00375n_1 + 0.005n_2 + 0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} \leq 1; \\
& \quad n_1 \geq 0, n_2 \geq 0;
\end{aligned} \tag{6.32}$$

Degree of Difficulty of the problem (6.32) is = (6-(2+1))=3

Hence the dual problem of the above problem (6.32) is given as:

$$\begin{aligned}
\max \quad & v(w_{0i}^*) = \left( (9802.9337/w_{01})^{w_{01}} \right) \times \left( (25885.0913/w_{02})^{w_{02}} \right) \times \\
& \left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0025}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.00325}{w_{14}} \right)^{w_{14}} \right) \\
& ((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
\text{subject to} \quad & w_{01} + w_{02} = 1; \quad (\text{normality condition}) \quad (ii) \\
& \left. \begin{aligned} -w_{01} + w_{11} + (1/2)w_{13} &= 0 \\ -w_{11} + w_{12} + (1/2)w_{14} &= 0 \end{aligned} \right\} \quad (\text{orthogonality condition}) \quad (iii) \\
& w_{01}, w_{02}, w_{11}, w_{12}, w_{13}, w_{14} \geq 0 \quad (\text{positivity condition}) \quad (iv)
\end{aligned} \quad (6.33)$$

For orthogonality condition defined in expression 6.33(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{13} = 0 \\ -w_{02} + w_{12} + (1/2)w_{14} = 0 \end{cases}$$

After solving the formulated dual problem (6.33) using lingo software we obtain the following values of the dual variables which are given as:

$$w_{01} = 0.3483659, w_{02} = 0.6516341 \text{ and } v(w_{0i}^*) = 323.3292$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem I are given as follows:

$$n_1^* = 87, n_2^* = 123 \text{ and the objective value of the primal problem is } 323.3292$$

**Example 2:** The following data are from Ghufraan *et al.* (2012). Table 2 gives the artificial data for four characteristics and four strata. The population size  $N$  is assumed to be 1000. This gives  $N_1 = 81$ ,  $N_2 = 343$ ,  $N_3 = 455$  and  $N_4 = 121$ . It is also assumed that the total budget of the survey  $C = 4500$  units with an overhead cost  $c_0 = 500$  units. Thus  $C_0 = (C - c_0) = 4,500 - 500 = 4,000$  units are available for measurements or measurements and traveling as the case may be. Using the values given in Table 2 the values of  $A_h$  are obtained as:

$$A_1 = 0.072830578, A_2 = 0.072830578, A_3 = 0.072830578 \text{ and } A_4 = 0.072830578.$$

**Table 2: Data for four Strata and four characteristics**

$h$	$W_h$	$\pi_{sh1}$	$\pi_{sh2}$	$\pi_{sh3}$	$\pi_{sh4}$	$M_h$	$P_h$	Travel cost is not <u>significant</u> $c_h$	Travel cost is <u>significant</u> $c'_h \quad t'_h$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)		
1	0.0808	0.28	0.33	0.40	0.62	0.80	0.70	25	15 10
2	0.3434	0.48	0.53	0.35	0.22	0.80	0.70	33	20 13
3	0.4546	0.68	0.73	0.55	0.82	0.80	0.70	40	30 10
4	0.1212	0.88	0.93	0.75	0.32	0.80	0.70	30	18 12

The increases in the variances for the individual characteristics are:

$$x_1 = 0.000001121883 \text{ and } x_2 = 0.000002463538.$$

For solving this MOGP problem, we shall first solve the two sub-problems such as:

**Sub problem1:** On substituting the table values in sub problem 1, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{01} &= \frac{0.001791658449}{n_1} + \frac{0.038022161}{n_2} + \frac{0.06002072}{n_3} + \frac{0.00262104527}{n_4} \\ \text{subject to} \quad &0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.34)$$

The dual of the above problem (6.34) is obtained as follows:

$$\begin{aligned}
 \max \quad v(w^*) &= \left( (0.001791658449/w_{01})^{w_{01}} \times (0.038022161/w_{02})^{w_{02}} \times \right. \\
 &\left. (0.06002072/w_{03})^{w_{03}} \times (0.00262104527/w_{04})^{w_{04}} \times \right. \\
 &\left. \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \times \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \times \left( \frac{0.01}{w_{13}} \right)^{w_{13}} \times \left( \frac{0.0075}{w_{14}} \right)^{w_{14}} \right) \times \\
 &((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
 \text{subject to} \quad & \left. \begin{aligned} &w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition) } (ii) \\ &\left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{02} + w_{12} &= 0 \\ -w_{03} + w_{13} &= 0 \\ -w_{04} + w_{14} &= 0 \end{aligned} \right\} \text{ (orthogonality condition) } (iii) \\ &\left. \begin{aligned} w_{01}, w_{02}, w_{03}, w_{04} &> 0 \\ w_{11}, w_{12}, w_{13}, w_{14} &\geq 0 \end{aligned} \right\} \text{ (positivity condition) } (iv) \end{aligned} \right\} \quad (6.35)
 \end{aligned}$$

For orthogonality condition defined in expression 6.35(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \\ -w_{03} + w_{13} = 0 \\ -w_{04} + w_{14} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06693948, w_{02} = 0.3542909, w_{03} = 0.04900780, w_{04} = 0.08869161$$

$$\text{and } v(w^*) = 0.002499026$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 11, n_2^* = 43, n_3^* = 49, n_4^* = 12 \text{ and the objective value of the primal problem is } 0.002499026.$$

**Sub problem2:** On substituting the table values in sub problem 2, we have obtained the expressions given below:

$$\left. \begin{aligned}
\min f_{02} &= \frac{0.001918966929}{n_1} + \frac{0.037963199}{n_2} \\
&\quad + \frac{0.055784166}{n_3} + \frac{0.00202612295}{n_4}, \\
\text{subject to} \\
&0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\
&n_1, n_2, n_3, n_4 \geq 0.
\end{aligned} \right\} \quad (6.36)$$

The dual of the above problem (6.36) is obtained as follows:

$$\left. \begin{aligned}
\max v(w^*) &= \left( (0.001918966929/w_{01})^{w_{01}} \right) \times \left( (0.037963199/w_{02})^{w_{02}} \right) \times \\
&\quad \left( (0.055784166/w_{03})^{w_{03}} \right) \times \left( (0.00202612295/w_{04})^{w_{04}} \right) \times \\
&\quad \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.01}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0075}{w_{14}} \right)^{w_{14}} \right) \times \\
&\quad ((w_{11} + w_{12} + w_{13} + w_{14})^{-(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
\text{subject to} \\
&w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii) \\
&\left. \begin{aligned}
&-w_{01} + w_{11} = 0 \\
&-w_{02} + w_{12} = 0 \\
&-w_{03} + w_{13} = 0 \\
&-w_{04} + w_{14} = 0
\end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
&w_{01}, w_{02}, w_{03}, w_{04} > 0; w_{11}, w_{12}, w_{13}, w_{14} \geq 0; \text{ (positivity condition)} \quad (iv)
\end{aligned} \right\} \quad (6.37)$$

For orthogonality condition defined in expression 6.37(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \\ -w_{03} + w_{13} = 0 \\ -w_{04} + w_{14} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$\begin{aligned}
w_{01} &= 0.07114580, w_{02} = 0.3635645, w_{03} = 0.4852073, w_{04} = 0.08008237 \\
\text{and } v(w^*) &= 0.002369488
\end{aligned}$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$n_1^* = 11, n_2^* = 44, n_3^* = 49, n_4^* = 11$  and the objective value of the primal problem is 0.002369488.

**Sub problem3:** On substituting the table values in sub problem 3, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{03} &= \frac{0.0020242358225}{n_1} + \frac{0.03541605}{n_2} \\ &+ \frac{0.066199888}{n_3} + \frac{0.003824110406}{n_4}, \\ \text{subject to} & \\ &0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.38)$$

The dual of the above problem (6.38) is obtained as follows:

$$\left. \begin{aligned} \max v(w_{0i}^*) &= \left( (0.0020242358225/w_{01})^{w_{01}} \right) \times \left( (0.03541605/w_{02})^{w_{02}} \right) \times \\ &\left( (0.066199888/w_{03})^{w_{03}} \right) \times \left( (0.003824110406/w_{04})^{w_{04}} \right) \times \\ &\left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.01}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0075}{w_{14}} \right)^{w_{14}} \right) \times \\ &((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\ \text{subject to} & \\ &w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii) \\ &\left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{02} + w_{12} &= 0 \\ -w_{03} + w_{13} &= 0 \\ -w_{04} + w_{14} &= 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\ &w_{01}, w_{02}, w_{03}, w_{04} > 0; w_{11}, w_{12}, w_{13}, w_{14} \geq 0; \text{ (positivity condition)} \quad (iv) \end{aligned} \right\} \quad (6.39)$$

For orthogonality condition defined in expression 6.39(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \\ -w_{03} + w_{13} = 0 \\ -w_{04} + w_{14} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06903756, w_{02} = 0.3303004, w_{03} = 0.4971778, w_{04} = 0.1034842$$

$$\text{and } v(w^*) = 0.002678157$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$n_1^* = 11, n_2^* = 40, n_3^* = 50, n_4^* = 14$  and the objective value of the primal problem is 0.002678157.

**Sub problem 4:** On substituting the table values in sub problem 2, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{04} &= \frac{0.002013632209}{n_1} + \frac{0.028824123}{n_2} \\ &+ \frac{0.045554438}{n_3} + \frac{0.00426626255}{n_4}, \\ \text{subject to} \\ 0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 &\leq 1, \\ n_1, n_2, n_3, n_4 &\geq 0. \end{aligned} \right\} \quad (6.40)$$



The dual of the above problem (6.40) is obtained as follows:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left\{ \left( (0.002013632209/w_{01})^{w_{01}} \right) \times \left( (0.028824123/w_{02})^{w_{02}} \right) \times \right. \\
 & \left. \left( (0.0045554438/w_{03})^{w_{03}} \right) \times \left( (0.00426626255/w_{04})^{w_{04}} \right) \times \right. \\
 & \left. \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.01}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0075}{w_{14}} \right)^{w_{14}} \right) \times \right. \\
 & \left. ((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \right. \quad (i) \\
 \text{subject to} \quad & \left. \begin{aligned} & w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii) \\ & \left. \begin{aligned} -w_{01} + w_{11} &= 0 \\ -w_{02} + w_{12} &= 0 \\ -w_{03} + w_{13} &= 0 \\ -w_{04} + w_{14} &= 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\ & w_{01}, w_{02}, w_{03}, w_{04} > 0; w_{11}, w_{12}, w_{13}, w_{14} \geq 0; \text{ (positivity condition)} \quad (iv) \end{aligned} \right\} \quad (6.41)
 \end{aligned}$$

For orthogonality condition defined in expression 6.41(iii) are evaluated with the help of the payoff matrix which is defined below:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \\ -w_{03} + w_{13} = 0 \\ -w_{04} + w_{14} = 0 \end{cases}$$

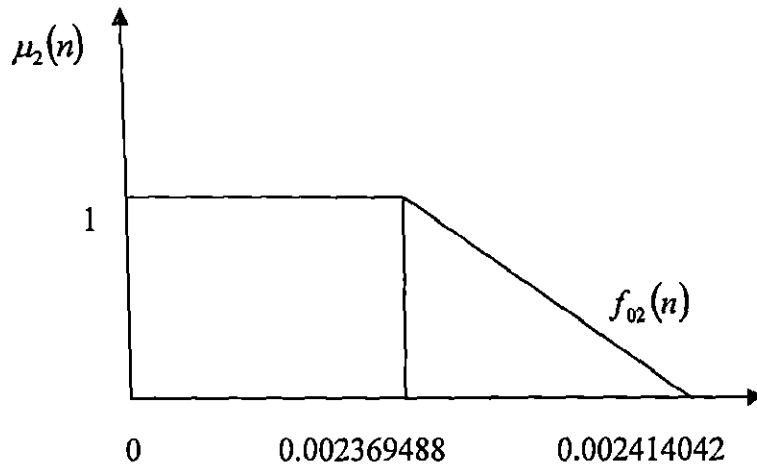
Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06903756, w_{02} = 0.3303004, w_{03} = 0.4971778, w_{04} = 0.1034842$$

$$\text{and } v(w^*) = 0.002678157$$

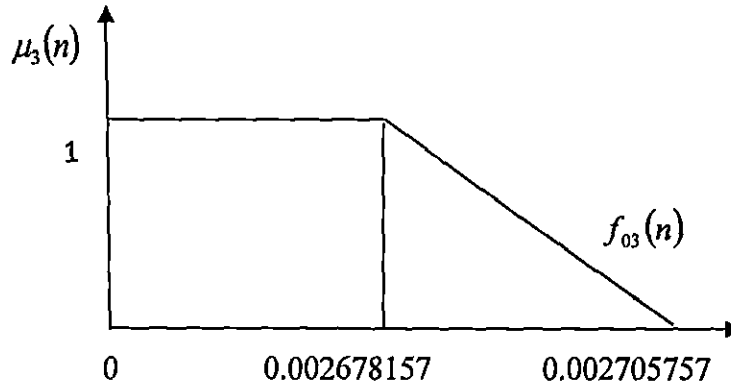
The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal-dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 12, n_2^* = 41, n_3^* = 46, n_4^* = 16 \text{ and the objective value of the primal problem is } 0.002113091.$$



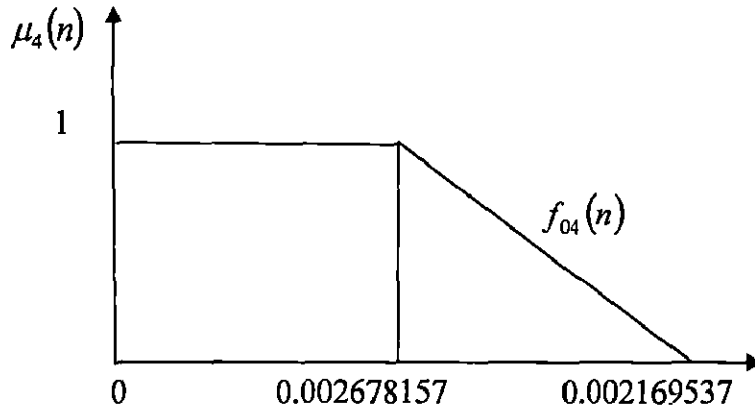
$$\mu_3(n) = \begin{cases} 1 & , \text{if } f_{03}(n) \leq 0.002678157 \\ \frac{0.002705757 - f_{03}(n)}{0.0000276} & , \text{if } 0.002678157 \leq f_{03}(n) \leq 0.002705757 \\ 0 & , \text{if } f_{03}(n) \geq 0.002705757 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_4(n) = \begin{cases} 1 & , \text{if } f_{04}(n) \leq 0.002113091 \\ \frac{0.00216953 - f_{04}(n)}{0.000056439} & , \text{if } 0.002678157 \leq f_{04}(n) \leq 0.002169537 \\ 0 & , \text{if } f_{04}(n) \geq 0.002169537 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_4(n)$



On applying the max-addition operator, the MOGPP reduces to the crisp problem as:

$$\begin{aligned}
 & \max (\mu_1(n) + \mu_2(n) + \mu_3(n) + \mu_4(n)) \\
 & \max \left\{ \frac{0.002532702 - f_{01}(n)}{0.000033676} + \frac{0.002414042 - f_{02}(n)}{0.000044554} + \right. \\
 & \left. \frac{0.002705757 - f_{03}(n)}{0.0000276} + \frac{0.002169537 - f_{04}(n)}{0.000056439} \right\} \\
 & \max \left\{ 265.8642 - \left( \frac{f_{01}(n)}{0.000033676} + \frac{f_{02}(n)}{0.000044554} + \frac{f_{03}(n)}{0.0000276} + \frac{f_{04}(n)}{0.000056439} \right) \right\} \\
 & \text{subject to} \\
 & \quad 0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\
 & \quad n_1, n_2, n_3, n_4 \geq 0,
 \end{aligned} \tag{6.42}$$

In order to maximize the above problem,

We have to minimize  $\left( \frac{f_{01}(n)}{0.000033676} + \frac{f_{02}(n)}{0.000044554} + \frac{f_{03}(n)}{0.0000276} + \frac{f_{04}(n)}{0.000056439} \right)$ ,

subject to the constraints as described below:

Now the pay-off matrix of the above problems is given below:

	$f_{01}(n)$	$f_{02}(n)$	$f_{03}(n)$	$f_{04}(n)$
$(n^{(1)})$	0.002499026	0.002499026	0.002373141	0.002149898
$(n^{(2)})$	0.002369489	0.002369488	0.002705757	0.00216953
$(n^{(3)})$	0.002509273	0.002390984	0.002678157	0.002127847
$(n^{(4)})$	0.002532702	0.002414042	0.002695356	0.002113091

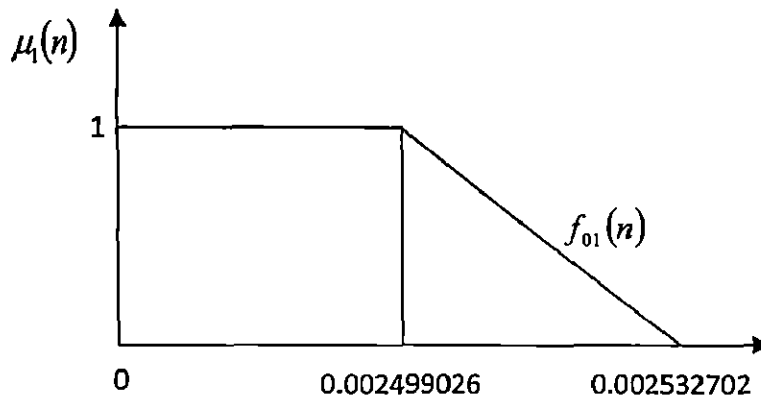
The lower and upper bonds of  $f_{01}(n)$ ,  $f_{02}(n)$ ,  $f_{03}(n)$  and  $f_{04}(n)$  can be obtained from the above pay-off matrix as follows:

$$0.002499026 \leq f_{01}(n) \leq 0.002532702, 0.002369488 \leq f_{02}(n) \leq 0.002414042, \\ 0.002678157 \leq f_{03}(n) \leq 0.002705757 \text{ and } 0.002113091 \leq f_{04}(n) \leq 0.00216953$$

Let  $\mu_1(n)$ ,  $\mu_2(n)$ ,  $\mu_3(n)$  and  $\mu_4(n)$  be the fuzzy membership function of the objective function  $f_{01}(n)$ ,  $f_{02}(n)$ ,  $f_{03}(n)$  and  $f_{04}(n)$  respectively and they are defined as:

$$\mu_1(n) = \begin{cases} 1 & , \text{if } f_{01}(n) \leq 0.002499026 \\ \frac{0.002532702 - f_{01}(n)}{0.000033676} & , \text{if } 0.002499026 \leq f_{01}(n) \leq 0.002532702 \\ 0 & , \text{if } f_{01}(n) \geq 0.002532702 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_2(n) = \begin{cases} 1 & , \text{if } f_{02}(n) \leq 0.002369488 \\ \frac{0.002414042 - f_{02}(n)}{0.000044554} & , \text{if } 0.002369488 \leq f_{02}(n) \leq 0.002414042 \\ 0 & , \text{if } f_{02}(n) \geq 0.002414042 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_2(n)$

$$\begin{aligned}
& \min \left( \frac{f_{01}(n)}{0.000033676} + \frac{f_{02}(n)}{0.000044554} + \frac{f_{03}(n)}{0.0000276} + \frac{f_{04}(n)}{0.000056439} \right) \\
& \quad \left( 29694.73809 \times f_{01}(n) + 22444.67388 \times f_{02}(n) + \right. \\
& \quad \left. 36231.88406 \times f_{03}(n) + 17718.24448 \times f_{04}(n) \right) \\
& \quad 29694.73809 \times \left( \frac{0.001791658449}{n_1} + \frac{0.038022161}{n_2} \right. \\
& \quad \left. + \frac{0.06002072}{n_3} + \frac{0.00262104527}{n_4} \right) \\
& \quad 22444.67388 \times \left( \frac{0.001918966929}{n_1} + \frac{0.037963199}{n_2} \right. \\
& \quad \left. + \frac{0.055784166}{n_3} + \frac{0.00202612295}{n_4} \right) \\
& \quad 36231.88406 \times \left( \frac{0.0020242358225}{n_1} + \frac{0.03541605}{n_2} \right. \\
& \quad \left. + \frac{0.066199888}{n_3} + \frac{0.003824110406}{n_4} \right) \\
& \quad 17718.24448 \times \left( \frac{0.002013632209}{n_1} + \frac{0.028824123}{n_2} \right. \\
& \quad \left. + \frac{0.045554438}{n_3} + \frac{0.00426626255}{n_4} \right) \\
& \quad \text{subject to } 0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\
& \quad \quad n_1, n_2, n_3, n_4 \geq 0,
\end{aligned} \tag{6.43}$$

On applying the max-addition operator, the final primal MOGPP with Linear cost function can be expressed as follows:

$$\begin{aligned}
& \min \left( \frac{205.2933}{n_1} + \frac{3775.0328}{n_2} + \frac{6240.0484}{n_3} + \frac{259.6211}{n_4} \right) \\
& \quad \text{subject to } \left. \begin{aligned} & 0.00625n_1 + 0.00825n_2 + 0.01n_3 + 0.0075n_4 \leq 1, \\ & n_1, n_2, n_3, n_4 \geq 0, \end{aligned} \right\}
\end{aligned} \tag{6.44}$$

Degree of Difficulty of the problem (6.44) is  $= (8 - (4 + 1)) = 3$ .

The dual of the above formulated problem (6.44) is given in Eqn. (6.45) as follows:

$$\begin{aligned}
& \max v(w_{0i}^*) = \left( (205.2933/w_{01})^{w_{01}} \right) \times \left( (3775.0328/w_{02})^{w_{02}} \right) \times \\
& \quad \left( (6240.0484/w_{03})^{w_{03}} \right) \times \left( (259.6211/w_{04})^{w_{04}} \right) \times \\
& \quad \left( \left( \frac{0.00625}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.00825}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.01}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0075}{w_{14}} \right)^{w_{14}} \right) \times \\
& \quad ((w_{11} + w_{12} + w_{13} + w_{14})^{(w_{11} + w_{12} + w_{13} + w_{14})}); \quad (i) \\
& \text{subject to} \\
& \quad w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii) \\
& \quad \left. \begin{aligned} & -w_{01} + w_{11} = 0 \\ & -w_{02} + w_{12} = 0 \\ & -w_{03} + w_{13} = 0 \\ & -w_{04} + w_{14} = 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
& \quad w_{01}, w_{02}, w_{03}, w_{04} > 0; w_{11}, w_{12}, w_{13}, w_{14} \geq 0; \text{ (positivity condition)} \quad (iv)
\end{aligned} \tag{6.45}$$

For orthogonality condition defined in expression 6.45(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} = 0 \\ -w_{02} + w_{12} = 0 \\ -w_{03} + w_{13} = 0 \\ -w_{04} + w_{14} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.07075944, w_{02} = 0.3486136, w_{03} = 0.4934589, w_{04} = 0.08716810$$

$$\text{and } v(w^*) = 256.2631$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 11, n_2^* = 42, n_3^* = 49, n_4^* = 12 \text{ and the objective value of the primal problem is } 256.2631.$$

**The MOGP problem with quadratic cost function:**

**Sub problem1:** On substituting the table values in sub problem 1, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{01} &= \frac{0.001791658449}{n_1} + \frac{0.038022161}{n_2} \\ &+ \frac{0.06002072}{n_3} + \frac{0.00262104527}{n_4}, \\ \text{subject to} \\ &0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\ &0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.46)$$

The dual of the above problem (6.46) is obtained as follows:

$$\left. \begin{aligned} \max v(w_{0i}^*) &= \left( (0.001791658449/w_{01})^{w_{01}} \right) \times \left( (0.038022161/w_{02})^{w_{02}} \right) \times \\ &\left( (0.06002072/w_{03})^{w_{03}} \right) \times \left( (0.00262104527/w_{04})^{w_{04}} \right) \times \\ &\left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0075}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0045}{w_{14}} \right)^{w_{14}} \right) \times \\ &\left( \left( \frac{0.0025}{w_{15}} \right)^{w_{15}} \right) \times \left( \left( \frac{0.00325}{w_{16}} \right)^{w_{16}} \right) \times \left( \left( \frac{0.0025}{w_{17}} \right)^{w_{17}} \right) \times \left( \left( \frac{0.003}{w_{18}} \right)^{w_{18}} \right) \\ &((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})^\wedge \\ &(w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})); \end{aligned} \right\} \quad (i) \quad (6.47)$$

$$\text{subject to } w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii)$$

$$\left. \begin{aligned} &-w_{01} + w_{11} + (1/2)w_{15} = 0 \\ &-w_{02} + w_{12} + (1/2)w_{16} = 0 \\ &-w_{03} + w_{13} + (1/2)w_{17} = 0 \\ &-w_{04} + w_{14} + (1/2)w_{18} = 0 \end{aligned} \right\} \text{ (orthogonality condition) } \quad (iii)$$

$$\left. \begin{aligned} &w_{01}, w_{02}, w_{03}, w_{04} > 0; \\ &w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18} \geq 0; \end{aligned} \right\} \text{ (positivity condition) } \quad (iv)$$

For orthogonality condition defined in expression 6.47(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \\ w_{15} \\ w_{16} \\ w_{17} \\ w_{18} \end{pmatrix}$$

$$= \begin{cases} -w_{01} + w_{11} + (1/2)w_{15} = 0 \\ -w_{02} + w_{12} + (1/2)w_{16} = 0 \\ -w_{03} + w_{13} + (1/2)w_{17} = 0 \\ -w_{04} + w_{14} + (1/2)w_{18} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06463467, w_{02} = 0.3367732, w_{03} = 0.5131199, w_{04} = 0.08547231$$

$$\text{and } v(w^*) = 0.001806518$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal-dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 15, n_2^* = 62, n_3^* = 65, n_4^* = 17 \text{ and the objective value of the primal problem is } 0.001806518.$$

**Sub problem2:** On substituting the table values in sub problem 2, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{02} &= \frac{0.001918966929}{n_1} + \frac{0.037963199}{n_2} \\ &+ \frac{0.055784166}{n_3} + \frac{0.00202612295}{n_4}, \\ \text{subject to} \\ &0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\ &0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.48)$$



The dual of the above problem (6.48) is obtained as follows:

$$\begin{aligned}
 \max \quad & v(w_{0i}^*) = \left( (0.001918966929/w_{01})^{w_{01}} \right) \times \left( (0.037963199/w_{02})^{w_{02}} \right) \times \\
 & \left( (0.055784166/w_{03})^{w_{03}} \right) \times \left( (0.00202612295/w_{04})^{w_{04}} \right) \times \\
 & \left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0075}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0045}{w_{14}} \right)^{w_{14}} \right) \times \\
 & \left( \left( \frac{0.0025}{w_{15}} \right)^{w_{15}} \right) \times \left( \left( \frac{0.00325}{w_{16}} \right)^{w_{16}} \right) \times \left( \left( \frac{0.0025}{w_{17}} \right)^{w_{17}} \right) \times \left( \left( \frac{0.003}{w_{18}} \right)^{w_{18}} \right) \\
 & ((w_{11}+w_{12}+w_{13}+w_{14} + w_{15}+w_{16}+w_{17}+w_{18})^\wedge \\
 & (w_{11}+w_{12}+w_{13}+w_{14} + w_{15}+w_{16}+w_{17}+w_{18})); \quad (i) \\
 \text{subject to} \quad & w_{01}+w_{02}+w_{03}+w_{04}=1; \text{ (normality condition)} \quad (ii) \\
 & \left. \begin{aligned} -w_{01} + w_{11} + (1/2)w_{15} &= 0 \\ -w_{02} + w_{12} + (1/2)w_{16} &= 0 \\ -w_{03} + w_{13} + (1/2)w_{17} &= 0 \\ -w_{04} + w_{14} + (1/2)w_{18} &= 0 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
 & \left. \begin{aligned} w_{01}, w_{02}, w_{03}, w_{04} &> 0; \\ w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18} &\geq 0; \end{aligned} \right\} \text{ (positivity condition)} \quad (iv)
 \end{aligned} \tag{6.49}$$

For orthogonality condition defined in expression 6.49(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \\ w_{15} \\ w_{16} \\ w_{17} \\ w_{18} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{15} = 0 \\ -w_{02} + w_{12} + (1/2)w_{16} = 0 \\ -w_{03} + w_{13} + (1/2)w_{17} = 0 \\ -w_{04} + w_{14} + (1/2)w_{18} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01}=0.06463467, w_{02}=0.3367732, w_{03}=0.5131199, w_{04}=0.08547231$$

$$\text{and } v(w^*) = 0.001711056$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$n_1^* = 16, n_2^* = 64, n_3^* = 64, n_4^* = 15$  and the objective value of the primal problem is 0.001711056.

**Sub problem3:** On substituting the table values in sub problem 3, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{03} &= \frac{0.0020242358225}{n_1} + \frac{0.03541605}{n_2} \\ &+ \frac{0.066199888}{n_3} + \frac{0.003824110406}{n_4}, \\ \text{subject to} \quad &0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\ &0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.50)$$

The dual of the above problem (50) is obtained as follows:

$$\left. \begin{aligned} \max v(w_{0i}^*) &= \left( (0.0020242358225/w_{01})^{w_{01}} \right) \times \left( (0.03541605/w_{02})^{w_{02}} \right) \times \\ &\left( (0.066199888/w_{03})^{w_{03}} \right) \times \left( (0.003824110406/w_{04})^{w_{04}} \right) \times \\ &\left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0075}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0045}{w_{14}} \right)^{w_{14}} \right) \times \\ &\left( \left( \frac{0.0025}{w_{15}} \right)^{w_{15}} \right) \times \left( \left( \frac{0.00325}{w_{16}} \right)^{w_{16}} \right) \times \left( \left( \frac{0.0025}{w_{17}} \right)^{w_{17}} \right) \times \left( \left( \frac{0.003}{w_{18}} \right)^{w_{18}} \right) \\ &((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})^{\wedge} \\ &(w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})); \end{aligned} \right\} \quad (6.51)$$

(i)

subject to  $w_{01} + w_{02} + w_{03} + w_{04} = 1$ ; (normality condition) (ii)

$$\left. \begin{aligned} &-w_{01} + w_{11} + (1/2)w_{15} = 0 \\ &-w_{02} + w_{12} + (1/2)w_{16} = 0 \\ &-w_{03} + w_{13} + (1/2)w_{17} = 0 \\ &-w_{04} + w_{14} + (1/2)w_{18} = 0 \end{aligned} \right\} \quad \text{(orthogonality condition)} \quad (iii)$$

$$\left. \begin{aligned} &w_{01}, w_{02}, w_{03}, w_{04} > 0; \\ &w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18} \geq 0; \end{aligned} \right\} \quad \text{(positivity condition)} \quad (iv)$$

For orthogonality condition defined in expression 6.51(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \\ w_{15} \\ w_{16} \\ w_{17} \\ w_{18} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{15} = 0 \\ -w_{02} + w_{12} + (1/2)w_{16} = 0 \\ -w_{03} + w_{13} + (1/2)w_{17} = 0 \\ -w_{04} + w_{14} + (1/2)w_{18} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06656883, w_{02} = 0.3139507, w_{03} = 0.5201115, w_{04} = 0.09936897$$

$$\text{and } v(w^*) = 0.001939421$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal-dual relationship theorem (6.11) as we have calculated in the sub-problem 1 are given as follows:

$$n_1^* = 16, n_2^* = 58, n_3^* = 66, n_4^* = 20 \text{ and the objective value of the primal problem is } 0.001939421.$$

**Sub problem 4:** On substituting the table values in sub problem 4, we have obtained the expressions given below:

$$\left. \begin{aligned} \min f_{04} &= \frac{0.002013632209}{n_1} + \frac{0.028824123}{n_2} \\ &\quad + \frac{0.045554438}{n_3} + \frac{0.00426626255}{n_4}, \\ \text{subject to } &0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\ &0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\ &n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.52)$$

The dual of the above problem (6.52) is obtained as follows:

$$\begin{aligned}
 \max v(w_{0i}^*) &= \left( (0.002013632209/w_{01})^{w_{01}} \right) \times \left( (0.028824123/w_{02})^{w_{02}} \right) \times \\
 &\left( (0.0045554438/w_{03})^{w_{03}} \right) \times \left( (0.00426626255/w_{04})^{w_{04}} \right) \times \\
 &\left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \times \left( \left( \frac{0.0075}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0045}{w_{14}} \right)^{w_{14}} \right) \times \\
 &\left( \left( \frac{0.0025}{w_{15}} \right)^{w_{15}} \right) \times \left( \left( \frac{0.00325}{w_{16}} \right)^{w_{16}} \right) \times \left( \left( \frac{0.0025}{w_{17}} \right)^{w_{17}} \right) \times \left( \left( \frac{0.003}{w_{18}} \right)^{w_{18}} \right) \\
 &((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})^\wedge \\
 &(w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})); \quad (i)
 \end{aligned}
 \tag{6.53}$$

subject to

$$\begin{aligned}
 &w_{01} + w_{02} + w_{03} + w_{04} = 1; \text{ (normality condition)} \quad (ii) \\
 &\left. \begin{aligned}
 &-w_{01} + w_{11} + (1/2)w_{15} = 0 \\
 &-w_{02} + w_{12} + (1/2)w_{16} = 0 \\
 &-w_{03} + w_{13} + (1/2)w_{17} = 0 \\
 &-w_{04} + w_{14} + (1/2)w_{18} = 0
 \end{aligned} \right\} \text{ (orthogonality condition)} \quad (iii) \\
 &\left. \begin{aligned}
 &w_{01}, w_{02}, w_{03}, w_{04} > 0; \\
 &w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18} \geq 0;
 \end{aligned} \right\} \text{ (positivity condition)} \quad (iv)
 \end{aligned}$$

For orthogonality condition defined in expression 6.53(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix}
 -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2)
 \end{pmatrix}
 \begin{pmatrix}
 w_{01} \\
 w_{02} \\
 w_{03} \\
 w_{04} \\
 w_{11} \\
 w_{12} \\
 w_{13} \\
 w_{14} \\
 w_{15} \\
 w_{16} \\
 w_{17} \\
 w_{18}
 \end{pmatrix}
 =
 \begin{cases}
 -w_{01} + w_{11} + (1/2)w_{15} = 0 \\
 -w_{02} + w_{12} + (1/2)w_{16} = 0 \\
 -w_{03} + w_{13} + (1/2)w_{17} = 0 \\
 -w_{04} + w_{14} + (1/2)w_{18} = 0
 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.07449375, w_{02} = 0.3198387, w_{03} = 0.4874748, w_{04} = 0.1181927$$

$$\text{and } v(w^*) = 0.001521403$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (11) as we have calculated in the sub-problem 1 are given as follows:

$n_1^* = 18$ ,  $n_2^* = 59$ ,  $n_3^* = 61$ ,  $n_4^* = 24$  and the objective value of the primal problem is 0.001521403 .

Now the pay-off matrix of the above problems is given below:

	$f_{01}(n)$	$f_{02}(n)$	$f_{03}(n)$	$f_{04}(n)$
$(n^{(1)})$	0.001806518	0.001713394	0.001947460	0.001547311
$(n^{(2)})$	0.001809149	0.001711096	0.001958840	0.001561365
$(n^{(3)})$	0.001813597	0.001726295	0.001939423	0.001531976
$(n^{(4)})$	0.001830362	0.001742483	0.001951784	0.001521403

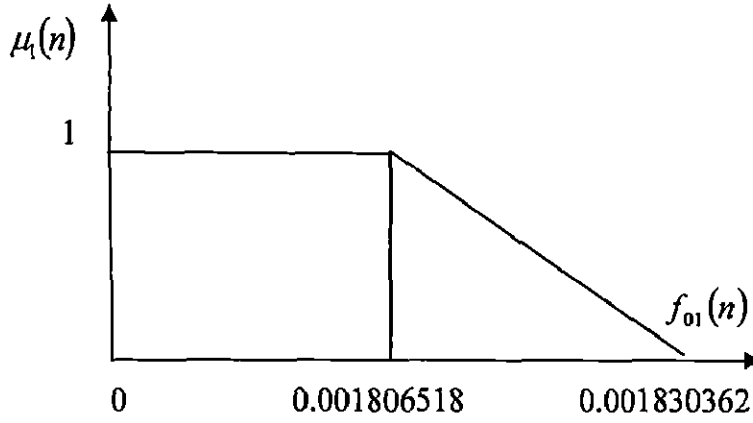
The lower and upper bonds of  $f_{01}(n)$ ,  $f_{02}(n)$ ,  $f_{03}(n)$  and  $f_{04}(n)$  can be obtained from the above pay-off matrix as follows:

$$\begin{aligned} 0.001806518 &\leq f_{01}(n) \leq 0.001830362, \\ 0.001711056 &\leq f_{02}(n) \leq 0.001742483, \\ 0.001939421 &\leq f_{03}(n) \leq 0.001958840 \\ \text{and } 0.001521403 &\leq f_{04}(n) \leq 0.001561365 \end{aligned}$$

Let  $\mu_1(n)$ ,  $\mu_2(n)$ ,  $\mu_3(n)$  and  $\mu_4(n)$  be the fuzzy membership function of the objective function  $f_{01}(n)$ ,  $f_{02}(n)$ ,  $f_{03}(n)$  and  $f_{04}(n)$  respectively and they are defined as:

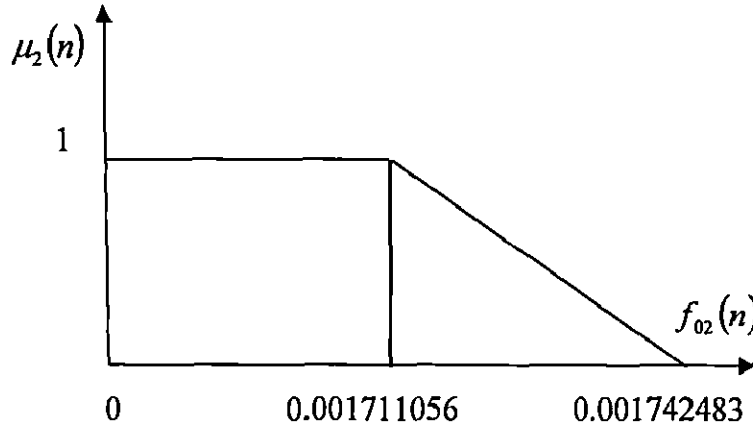
$$\mu_1(n) = \begin{cases} 1 & , \text{if } f_{01}(n) \leq 0.001806518 \\ \frac{0.001830362 - f_{01}(n)}{0.000023844} & , \text{if } 0.001806518 \leq f_{01}(n) \leq 0.001830362 \\ 0 & , \text{if } f_{01}(n) \geq 0.001830362 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



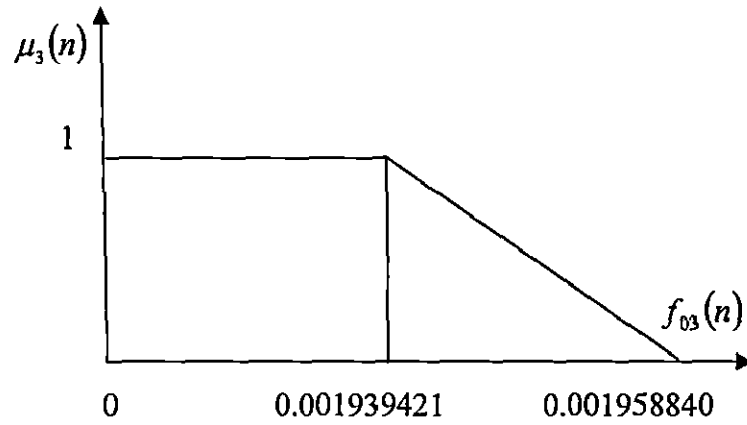
$$\mu_2(n) = \begin{cases} 1 & , \text{if } f_{02}(n) \leq 0.001711056 \\ \frac{0.001742483 - f_{02}(n)}{0.000031427} & , \text{if } 0.001711056 \leq f_{02}(n) \leq 0.001742483 \\ 0 & , \text{if } f_{02}(n) \geq 0.001742483 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



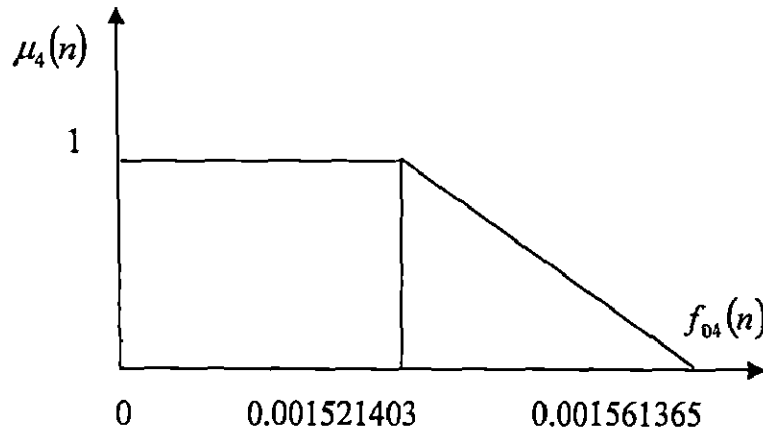
$$\mu_3(n) = \begin{cases} 1 & , \text{if } f_{03}(n) \leq 0.001939421 \\ \frac{0.001958840 - f_{03}(n)}{0.000019419} & , \text{if } 0.001939421 \leq f_{03}(n) \leq 0.001958840 \\ 0 & , \text{if } f_{03}(n) \geq 0.001958840 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



$$\mu_4(n) = \begin{cases} 1 & , \text{if } f_{04}(n) \leq 0.001521403 \\ \frac{0.001561365 - f_{04}(n)}{0.000039962} & , \text{if } 0.001521403 \leq f_{04}(n) \leq 0.001561365 \\ 0 & , \text{if } f_{04}(n) \geq 0.001561365 \end{cases}$$

The following figure illustrated the graph of the fuzzy membership function  $\mu_1(n)$



On applying the max-addition operator, the MOGPP reduces to the crisp problem as:

$$\begin{aligned}
 & \max \left( \mu_1(n) + \mu_2(n) + \mu_3(n) + \mu_4(n) \right) \\
 & \max \left\{ \frac{0.001830362 - f_{01}(n)}{0.000023844} + \frac{0.001742483 - f_{02}(n)}{0.000031427} + \right. \\
 & \left. \frac{0.001958840 - f_{03}(n)}{0.000019419} + \frac{0.001561365 - f_{04}(n)}{0.000039962} \right\} \\
 & \max \left\{ 272.15305 - \left( \frac{f_{01}(n)}{0.000023844} + \frac{f_{02}(n)}{0.000031427} + \right. \right. \\
 & \left. \left. \frac{f_{03}(n)}{0.000019419} + \frac{f_{04}(n)}{0.000039962} \right) \right\} \\
 & \text{subject to } 0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\
 & 0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\
 & n_1, n_2, n_3, n_4 \geq 0.
 \end{aligned} \tag{6.54}$$

In order to maximize the above problem,

we have to minimize  $\left( \frac{f_{01}(n)}{0.000023844} + \frac{f_{02}(n)}{0.000031427} + \frac{f_{03}(n)}{0.000019419} + \frac{f_{04}(n)}{0.000039962} \right),$

subject to the constraints as described below:



$$\begin{aligned}
\min \left\{ \begin{aligned} & \left( \frac{f_{01}(n)}{0.000023844} + \frac{f_{02}(n)}{0.000031427} + \frac{f_{03}(n)}{0.000019419} + \frac{f_{04}(n)}{0.000039962} \right) \\ & \left( 41939.27193 \times f_{01}(n) + 31819.677281 \times f_{02}(n) + \right. \\ & \quad \left. 51495.95757 \times f_{03}(n) + 25023.77258 \times f_{04}(n) \right) \\ & 41939.27193 \times \left( \frac{0.001791658449}{n_1} + \frac{0.038022161}{n_2} + \right. \\ & \quad \left. \frac{0.06002072}{n_3} + \frac{0.00262104527}{n_4} \right) \\ & 31819.677281 \times \left( \frac{0.001918966929}{n_1} + \frac{0.037963199}{n_2} + \right. \\ & \quad \left. \frac{0.055784166}{n_3} + \frac{0.00202612295}{n_4} \right) \\ & 51495.95757 \times \left( \frac{0.0020242358225}{n_1} + \frac{0.03541605}{n_2} + \right. \\ & \quad \left. \frac{0.066199888}{n_3} + \frac{0.003824110406}{n_4} \right) \\ & 25023.77258 \times \left( \frac{0.002013632209}{n_1} + \frac{0.028824123}{n_2} + \right. \\ & \quad \left. \frac{0.045554438}{n_3} + \frac{0.00426626255}{n_4} \right) \end{aligned} \right\} \quad (6.55) \\
\text{subject to } 0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\
0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\
n_1, n_2, n_3, n_4 \geq 0.
\end{aligned}$$

On applying the max-addition operator, the final primal MOGPP with cost function quadratic in  $\sqrt{n_h}$  and significant travel cost can be as follows:

$$\begin{aligned}
\min \left\{ \begin{aligned} & \frac{291.8525}{n_1} + \frac{5349.4011}{n_2} + \frac{8843.8486}{n_3} + \frac{478.182}{n_4} \\ & \text{subject to } 0.00375n_1 + 0.005n_2 + 0.0075n_3 + 0.0045n_4 + \\ & \quad 0.0025\sqrt{n_1} + 0.00325\sqrt{n_2} + 0.0025\sqrt{n_3} + 0.003\sqrt{n_4} \leq 1, \\ & \quad n_1, n_2, n_3, n_4 \geq 0. \end{aligned} \right\} \quad (6.56)
\end{aligned}$$

Degree of Difficulty of the problem (6.56) is  $= (12 - (4 + 1)) = 7$ .

The dual of the above final formulated problem (6.56) is given in Eqn. (6.57) as follows:

$$\begin{aligned}
\max v(w_{0i}^*) = & \left( (291.8525/w_{01})^{w_{01}} \right) \times \left( (5349.4011/w_{02})^{w_{02}} \right) \times \left( (8843.8486/w_{03})^{w_{03}} \right) \\
& \times \left( (478.182/w_{04})^{w_{04}} \right) \times \left( \left( \frac{0.00375}{w_{11}} \right)^{w_{11}} \right) \times \left( \left( \frac{0.005}{w_{12}} \right)^{w_{12}} \right) \\
& \times \left( \left( \frac{0.0075}{w_{13}} \right)^{w_{13}} \right) \times \left( \left( \frac{0.0045}{w_{14}} \right)^{w_{14}} \right) \times \left( \left( \frac{0.0025}{w_{15}} \right)^{w_{15}} \right) \\
& \times \left( \left( \frac{0.00325}{w_{16}} \right)^{w_{16}} \right) \times \left( \left( \frac{0.0025}{w_{17}} \right)^{w_{17}} \right) \times \left( \left( \frac{0.003}{w_{18}} \right)^{w_{18}} \right) \\
& ((w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})^\wedge \\
& (w_{11} + w_{12} + w_{13} + w_{14} + w_{15} + w_{16} + w_{17} + w_{18})); \quad (i)
\end{aligned} \tag{6.57}$$

subject to

$$\begin{aligned}
& w_{01} + w_{02} + w_{03} + w_{04} = 1; \quad (\text{normality condition}) \quad (ii) \\
& \left. \begin{aligned}
& -w_{01} + w_{11} + (1/2)w_{15} = 0 \\
& -w_{02} + w_{12} + (1/2)w_{16} = 0 \\
& -w_{03} + w_{13} + (1/2)w_{17} = 0 \\
& -w_{04} + w_{14} + (1/2)w_{18} = 0
\end{aligned} \right\} (\text{orthogonality condition}) \quad (iii) \\
& \left. \begin{aligned}
& w_{01}, w_{02}, w_{03}, w_{04} > 0 \\
& w_{11}, w_{12}, w_{13}, w_{14}, w_{15}, w_{16}, w_{17}, w_{18} \leq 0
\end{aligned} \right\} (\text{positivity condition}) \quad (iv)
\end{aligned}$$

For orthogonality condition defined in expression 6.57(iii) are evaluated with the help of the payoff matrix which is defined below

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (1/2) \end{pmatrix} \begin{pmatrix} w_{01} \\ w_{02} \\ w_{03} \\ w_{04} \\ w_{11} \\ w_{12} \\ w_{13} \\ w_{14} \\ w_{15} \\ w_{16} \\ w_{17} \\ w_{18} \end{pmatrix} = \begin{cases} -w_{01} + w_{11} + (1/2)w_{15} = 0 \\ -w_{02} + w_{12} + (1/2)w_{16} = 0 \\ -w_{03} + w_{13} + (1/2)w_{17} = 0 \\ -w_{04} + w_{14} + (1/2)w_{18} = 0 \end{cases}$$

Solving the above formulated dual problems, we have the corresponding solution as:

$$w_{01} = 0.06755195, w_{02} = 0.327428, w_{03} = 0.5105126, w_{04} = 0.09444265 \text{ and } v(w^*) = 268.9892.$$

The optimal values  $n_h^*$  of the sample sizes of the primal problems can be calculated with the help of the primal–dual relationship theorem (6.11) as we have calculated in the sub-problem I are given as follows:

$$n_1^* = 16, n_2^* = 61, n_3^* = 64, n_4^* = 19 \text{ and the objective value of the primal problem is } 268.9892.$$

### **Conclusion:**

In this chapter the problem is formulated as a multi-objective nonlinear programming problem. A complete method of solution of the formulated problem is projected to solve the problem. A numerical example is worked out to illustrate the computational details of the proposed method. In the next part of this chapter the two-stage stratified Warner's randomized response (RR) model with travel cost is considered and fuzzy geometric programming approach is used to obtain the optimum allocations of sample sizes. The chances of non-response in a multivariate stratified sample survey when the sampling is done with the sensitive questions may be significantly high. To ascertain the practical utility of the fuzzy geometric programming approach in sample surveys problem of randomized response model is extended for multiple sensitive questions to illustrate the fuzzy geometric programming procedure.

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